Discrete Mathematics
Chapter 2 : Set Theory

GATE CS Lectures
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Section1: Engineering Mathematics

- **Discrete Mathematics**: Propositional and first order logic. Sets, relations, functions, partial orders and lattices. Monoids, Groups. Graphs: connectivity, matching, coloring. Combinatorics: counting, recurrence relations , generating functions.
- **Linear Algebra**: Matrices, determinants, system of linear equations, eigenvalues and eigenvectors, LU decomposition.
- **Calculus**: Limits, continuity and differentiability. Maxima and minima. Mean value theorem. Integration.
- **Probability and Statistics**: Random variables. Uniform, normal, exponential, poisson and binomial distributions. Mean, median, mode and standard deviation. Conditional probability and Bayes theorem.

- **Discrete Mathematics**: Propositional and first order logic. Sets, relations, functions, partial orders and lattices.Monoids, Groups.Graphs: connectivity, matching, coloring.Combinatorics : counting, recurrence relations , generating functions.
- Chapter 1: Logic
- Propositional Logic, Propositional Equivalences , Predicates and Quantifiers , Nested Quantifiers , Rules of Inference , Introduction to Proofs.
- Chapter 2 : Set Theory
- Sets, relations, functions, partial orders and lattices.Monoids, Groups.
- Chapter 3 : Graph Theory
- Chapter 4 : Combinatorics

Chapter 2 : Set Theory

- 2.1 Sets
- 2.2 Set Operations
- 2.3 Functions
- 2.4 Sequences and Summations
- 2.5 Cardinality of Sets
- 2.6 Relations and Their Properties
- 2.7 n-ary Relations and Their Applications
- 2.8 Representing Relations
- 2.9 Closures of Relations
- 2.10 Equivalence Relations
- 2.11 Partial Orderings
- 2.12 Groups

2.1 Sets

- **DEFINITION 1** A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.
- It is common for sets to be denoted using uppercase letters. Lowercase letters are usually used to denote elements of sets.
- There are several ways to describe a set. One way is to list all the members of a set, For example, the notation $\{a, b, c, d\}$ represents the set with the four elements a, b, c, and d. This way of describing a set is known as the **roster method**.
- **EXAMPLE 1** The set V of all vowels in the English alphabet can be written as $V = \{a,e,i,o,u\}$.
- **EXAMPLE 2** The set O of odd positive integers less than 10 can be expressed by $O = \{1,3,5,7,9\}$.
- **EXAMPLE 3** The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$.
- Another way to describe a set is to use **set builder** notation.
- We characterize all those elements in the set by stating the property or properties.
- For instance, the set O of all odd positive integers less than 10 can be written as
- $O = \{x \mid x$ is an odd positive integer less than $10\}$, or, specifying the universe as the set of positive integers, as $O = \{x \in \mathbf{Z}^+ \mid x$ is odd and $x < 10\}$.

- We often use this type of notation to describe sets when it is impossible to list all the elements of the set.
- **N** = {0, 1, 2, 3, . . .}, the set of **natural numbers**
- **Z** = {. . . , -2, -1, 0, 1, 2, . . .}, the set of **integers**
- $\mathbf{Z}^+$ = {1, 2, 3, . . .}, the set of **positive integers**
- **Q** = $\{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**
- **R**, the set of **real numbers**
- $\mathbf{R}^+$, the set of **positive real numbers**
- **C**, the set of **complex numbers**.
- **Intervals:** When *a* and *b* are real numbers with $a < b$, we write $[a, b] = \{x \mid a \leq x \leq b\}$
- $[a, b) = \{x \mid a \leq x < b\}$ $(a, b] = \{x \mid a < x \leq b\}$ $(a, b) = \{x \mid a < x < b\}$
- $[a, b]$ is called the **closed interval** from *a* to *b* & (a, b) is called the **open interval** from *a* to *b*.
- Sets can have other sets as members.
- **EXAMPLE 4** The set {**N**, **Z**, **Q**, **R**} is a set containing four elements, each of which is a set.
- **DEFINITION 2** Two sets are *equal* if and only if they have the same elements . Therefore, if *A* and *B* are sets,then *A* and *B* are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if *A* and *B* are equal sets.

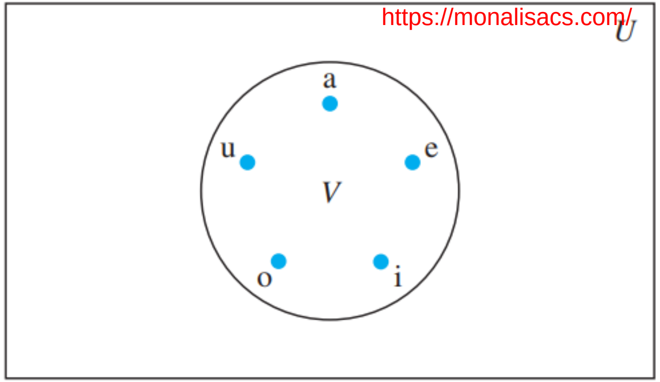
- **EXAMPLE 5** The sets {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so {1, 3, 3, 3, 5, 5, 5, 5} is the same as the set {1, 3, 5} because they have the same elements.

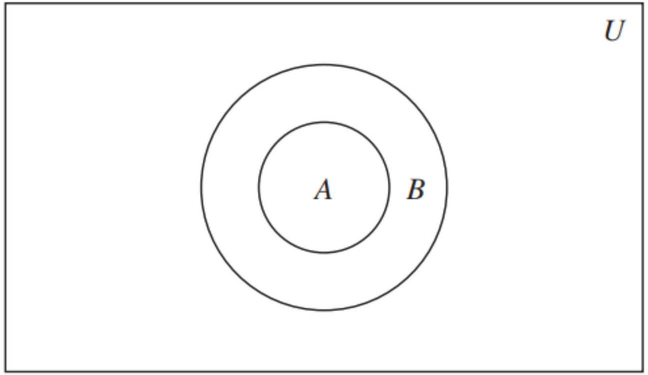
- **THE EMPTY SET** There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by ∅. The empty set can also be denoted by { }.
- A set with one element is called a **singleton set**.
- A common error is to confuse the empty set ∅ with the set {∅}, which is a singleton set.

- **Venn Diagrams**
- Sets can be represented graphically using Venn diagrams, named after the English mathematician John Venn, who introduced their use in 1881.
- In Venn diagrams the **universal set** *U*, which contains all the objects under consideration, is represented by a rectangle.
- Inside this rectangle, circles or other geometrical figures are used to represent sets.
- Sometimes points are used to represent the particular elements of the set.
- Venn diagrams are often used to indicate the relationships between sets.
- **EXAMPLE 6** Draw a Venn diagram that represents *V*, the set of vowels in the English alphabet.

Subsets

- **DEFINITION 3** The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.
- *Showing that A is a Subset of B* To show that $A \subseteq B$, show that if $x \in A$ then $x \in B$.
- *Showing that A is Not a Subset of B* To show that $A \nsubseteq B$, find a single $x \in A$ such that $x \notin B$.
- **EXAMPLE 7** The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10, the set of rational numbers is a subset of the set of real numbers.
- **Venn Diagram Showing that *A* Is a Subset of *B***
- **THEOREM 1** For every set S, (*i*) $\emptyset \subseteq S$ and (*ii*) $S \subseteq S$.
- ***Proof:*** To show that $\emptyset \subseteq S$, we must show that $\forall x(x \in \emptyset \to x \in S)$ is true.
- Because the empty set contains no elements, it follows that $x \in \emptyset$ is always false.
- $x \in \emptyset \to x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true.





- Therefore, $\forall x(x \in \emptyset \to x \in S)$ is true.
- Note that this is an example of a vacuous proof.
- When we wish to emphasize that a set A is a subset of a set B but that $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B.
- A is a proper subset of B if and only if $\forall x(x \in A \to x \in B) \wedge \exists x(x \in B \wedge x \notin A)$
- *Showing Two Sets are Equal* To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.
- Sets may have other sets as members. For instance, we have the sets
- $A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $B = \{x \mid x$ is a subset of the set $\{a, b\}\}$.
- These two sets are equal, that is, $A = B$. Also note that $\{a\} \in A$, but $a \notin A$.

The Size of a Set

- **DEFINITION 4** Let S be a set. If there are exactly n distinct elements in S,we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.
- ***Remark:*** The term *cardinality* comes from the common usage of the term *cardinal number* as the size of a finite set.
- **EXAMPLE 8** Let A be the set of odd positive integers < 10. Then $|A| = 5$.
- **EXAMPLE 9** Let S be the set of letters in the English alphabet. $|S| = 26$.
- **EXAMPLE 10** Because the null set has no elements, it follows that $|\emptyset| = 0$.

- **DEFINITION 5** A set is said to be *infinite* if it is not finite.
- **EXAMPLE 11** The set of positive integers is infinite.

Power Sets

- **DEFINITION 6** Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $P(S)$
- **EXAMPLE 12** What is the power set of the set $\{0, 1, 2\}$?
- *Solution:* $P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$.
- **EXAMPLE 13** What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?
- *Solution:* The empty set has exactly one subset, namely, itself.
- Consequently , $P(\emptyset) = \{\emptyset\}$.
- The set $\{\emptyset\}$ has exactly two subsets, namely, $\emptyset$ and the set $\{\emptyset\}$ itself. $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.
- If a set has n elements, then its power set has 2^n elements.

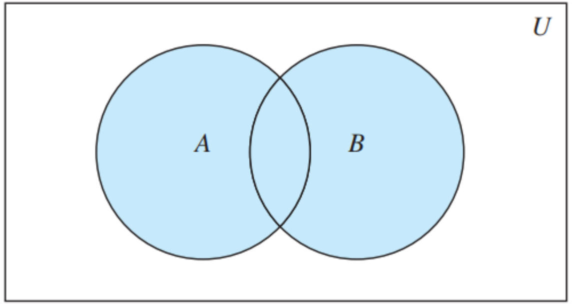
Cartesian Products

- **DEFINITION 7** The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, . . . , and a_n as its nth element
- Two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal.

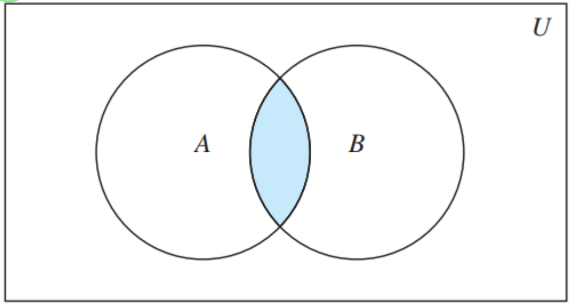
- In other words, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$, for $i = 1, 2, \ldots, n$.
- In particular, ordered 2-tuples are called **ordered pairs**.
- The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.
- Note that (a, b) and (b, a) are not equal unless $a = b$
- **DEFINITION 8** Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence, $A \times B = \{(a, b) | a \in A \land b \in B\}$.
- **EXAMPLE 14** What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?
- *Solution:* $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.
- $A \times B$ and $B \times A$ are not equal, unless $A = \emptyset$ or $B = \emptyset$ (so that $A \times B = \emptyset$) or $A = B$.
- **DEFINITION 9** The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words, $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$.
- **EXAMPLE 15** What is the Cartesian product $A \times B \times C$, where $A = \{0,1\}$, $B = \{1,2\}$, and $C = \{0,1,2\}$?
- *Solution:* $A \times B \times C$ consists of all ordered triples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Hence, $A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$.
- We use the notation A^2 to denote $A \times A$, the Cartesian product of the set A with itself. Similarly, $A^3 = A \times A \times A$, $A^4 = A \times A \times A \times A$, and so on.
- **EXAMPLE 16** Suppose that $A = \{1, 2\}$. $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ and $A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$.

2.2 Set Operations

- **DEFINITION 1** Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.
- $A \cup B = \{x \mid x \in A \vee x \in B\}$.
- **EXAMPLE 1** $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$.
- **DEFINITION 2** Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.
- $A \cap B = \{x \mid x \in A \wedge x \in B\}$.
- **EXAMPLE 2** $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$.
- **DEFINITION 3** Two sets are called *disjoint* if their intersection is the empty set.
- **EXAMPLE 3** Let $A=\{1,3,5,7,9\}$ and $B=\{2,4,6,8,10\}$. Because $A \cap B=\emptyset$, A and B are disjoint.

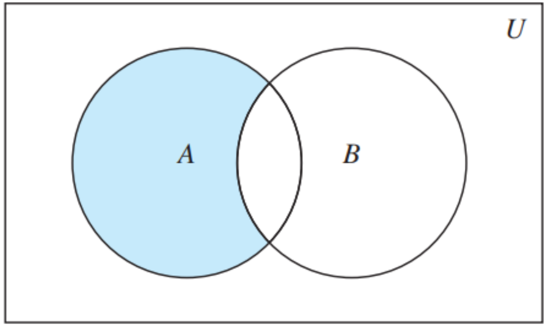


$A \cup B$ is shaded.



$A \cap B$ is shaded.

- **Principle of inclusion–exclusion.**
- $|A| + |B|$ counts each element that is in A but not in B or in B but not in A exactly once, and each element that is in both A and B exactly twice.
- Thus, if the number of elements that are in both A and B is subtracted from $|A| + |B|$, elements in $A \cap B$ will be counted only once. Hence,
- $|A \cup B| = |A| + |B| - |A \cap B|$.
- **DEFINITION 4** Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*.
- The difference of sets A and B is sometimes denoted by $A\backslash B$. $A - B = \{x \mid x \in A \wedge x \notin B\}$.
- **EXAMPLE 4** The difference of {1,3,5} and {1,2,3} is the set {5}; that is, {1,3,5} - {1,2,3}= {5}.
- **DEFINITION 5** Let U be the universal set. The *complement* of the set A, denoted by $\bar{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.



$A - B$ is shaded.

- $\bar{A} = \{x \mid x \in U \mid x \notin A\}$
- $A - B = A \cap \bar{B}$
- **EXAMPLE 5** Let $A = \{a, e, i, o, u\}$,U=All Alphabet
- $\bar{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$.

Table 1 : Set Identities

Identity	**Name**
$A \cap U = A$, $A \cup \emptyset = A$	Identity laws
$A \cup U = U$, $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$, $A \cap A = A$	Idempotent laws
$\overline{(\bar{A})} = A$	Complementation law
$A \cup B = B \cup A$, $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \bar{A} \cup \bar{B}$, $\overline{A \cup B} = \bar{A} \cap \bar{B}$	De Morgan’s laws
$A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$	Absorption laws
$A \cup \bar{A} = U$, $A \cap \bar{A} = \emptyset$	Complement laws

- **EXAMPLE 6** Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \bar{A} \cup \bar{B}$.
- *Solution:* We can prove this identity with the following steps.
- $\overline{A \cap B} = \{x \mid x \notin A \cap B\}$ by definition of complement
- $= \{x \mid \neg(x \in (A \cap B))\}$ by definition of does not belong symbol
- $= \{x \mid \neg(x \in A \wedge x \in B)\}$ by definition of intersection
- $= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$ by the first De Morgan law for logical equivalences
- $= \{x \mid x \notin A \vee x \notin B\}$ by definition of does not belong symbol
- $= \{x \mid x \in \bar{A} \vee x \in \bar{B}\}$ by definition of complement
- $= \{x \mid x \in \bar{A} \cup \bar{B}\}$ by definition of union
- $= \bar{A} \cup \bar{B}$ by meaning of set builder notation
- **EXAMPLE 7**
- Let A, B, and C be sets. Show that $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$.
- *Solution:* $\overline{A \cup (B \cap C)} = \bar{A} \cap \overline{(B \cap C)}$ by the first De Morgan law
- $= \bar{A} \cap (\bar{B} \cup \bar{C})$ by the second De Morgan law
- $= (\bar{B} \cup \bar{C}) \cap \bar{A} = (\bar{C} \cup \bar{B}) \cap \bar{A}$ by the commutative law.
- Set identities can also be proved using **membership tables**.

- **EXAMPLE 8** Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- *Solution:* Because the columns for $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are the same, the identity is valid.

TABLE 2 **A Membership Table for the Distributive Property.**

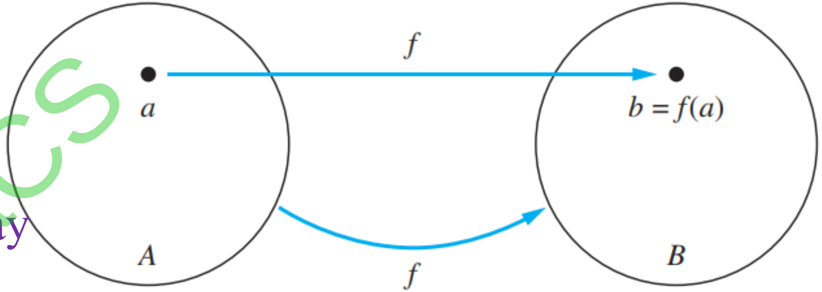
A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

- **Generalized Unions and Intersections**
- **EXAMPLE 9** Let $A = \{0,2,4,6,8\}$, $B = \{0,1,2,3,4\}$, and $C = \{0,3,6,9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?
- *Solution:* $A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}$.
- $A \cap B \cap C = \{0\}$.

- **DEFINITION 6** The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.
- We use the notation $A_1 \cup A_2 \cup \cdot \cdot \cdot \cup A_n = \bigcup_{i=1}^{n} A_i$ to denote the union of the sets $A_1, A_2, \ldots, A_n$.
- **DEFINITION 7** The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.
- We use the notation $A_1 \cap A_2 \cap \cdot \cdot \cdot \cap A_n = \bigcap_{i=1}^{n} A_i$ to denote the intersection of the sets A_1, $A_2, \ldots, A_n$.

2.3 Functions

- **DEFINITION 1** Let *A* and *B* be nonempty sets. A *function f* from *A* to *B* is an assignment of exactly one element of *B* to each element of *A*. We write *f (a) = b* if *b* is the unique element of *B* assigned by the function *f* to the element *a* of *A*. If *f* is a function from *A* to *B*, we write $f : A \to B$.
- Functions are sometimes also called **mappings** or **transformations**.
- **The Function *f* Maps *A* to *B*.**

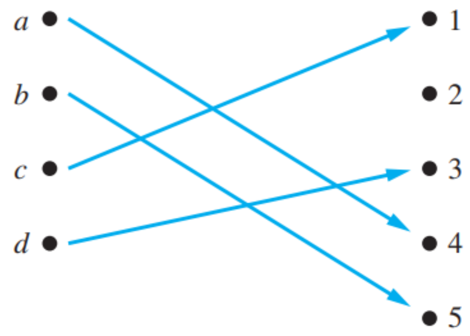


- **DEFINITION 2** If *f* is a function from *A* to *B*, we say that *A* is the *domain* of *f* and *B* is the *codomain* of *f*. If *f (a) = b*, we say that *b* is the *image* of *a* and *a* is a *preimage* of *b*. The *range*, or *image*, of *f* is the set of all images of elements of *A*. If *f* is a function from *A* to *B*, we say that *f maps A* to *B*.
- Two functions are **equal** when they have the same domain, have the same codomain, and map each element of their common domain to the same element in their common codomain .
- Number of Function possible from A to B= $|B|^{|A|}$
- **EXAMPLE 1** Let $f : Z \to Z$ assign the square of an integer to this integer. Then, $f(x) = x^2$, where the domain of f is the set of all integers, the codomain of f is the set of all integers that are perfect squares, namely, {0, 1, 4, 9, . . . }.

- **DEFINITION 3** Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and f_1f_2 are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by
- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$,
- $(f_1f_2)(x) = f_1(x)f_2(x)$.
- **EXAMPLE 2** Let f_1 and f_2 be functions from $\mathbf{R}$ to $\mathbf{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and f_1f_2?
- *Solution:* From the definition of the sum and product of functions, it follows that
- $(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$
- $(f_1f_2)(x) = x^2(x - x^2) = x^3 - x^4$.
- **DEFINITION 4** Let f be a function from A to B and let S be a subset of A. The *image* of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so $f(S) = \{t \mid \exists s \in S\ (t = f(s))\}$.
- We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.
- **EXAMPLE 3** Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$.
- The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

One-to-One and Onto Functions

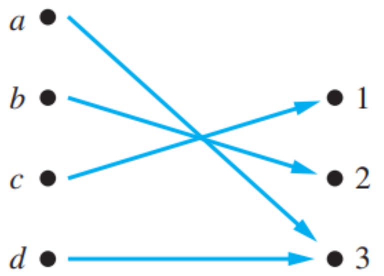
- Some functions never assign the same value to two different domain elements. These functions are said to be **one-to-one**.
- **DEFINITION 5** A function *f* is said to be *one-to-one*, or an *injunction*, if and only if $f(a) = f(b)$ implies that $a = b$ for all *a* and *b* in the domain of *f*. A function is said to be *injective* if it is one-to-one
- One to one function A→B possible if $|A| \leq |B|$
- Number of one to one functions possible A→B = $^{|B|}P_{|A|}$
- If $|A|=|B|=n$ then number of one to one functions possible n!
- **EXAMPLE 4** Determine whether the function *f* from {*a*, *b*, *c*, *d*} to {1, 2, 3, 4, 5} with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.



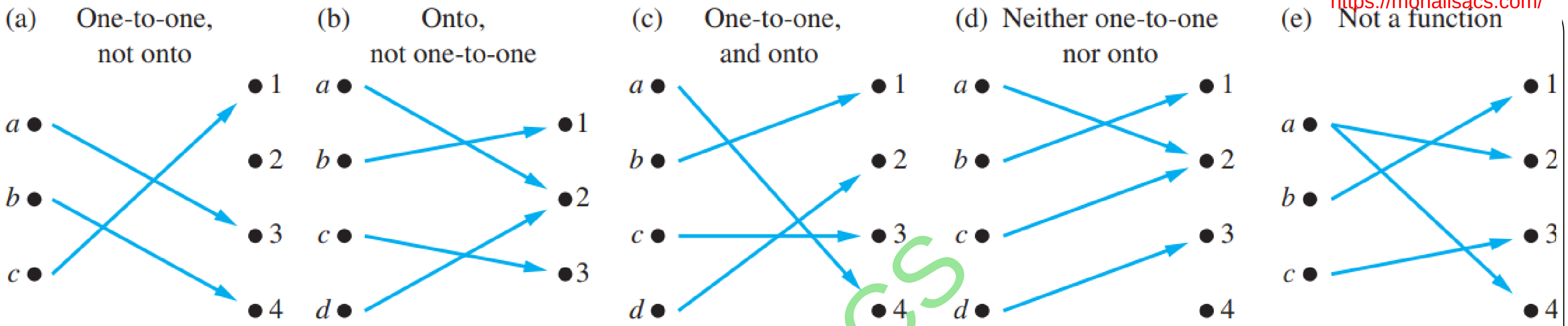
- *Solution:* The function *f* is one-to-one because *f* takes on different values at domain.
- **EXAMPLE 5** Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.
- *Solution:* The function $f(x) = x^2$ is not one-to-one because, for instance, $f(1) = f(-1) = 1$, but $1 \neq -1$.
- The function $f(x) = x^2$ with its domain restricted to $\mathbf{Z}^+$ is one-to-one.

- **EXAMPLE 6** Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to one.
- *Solution:* The function $f(x) = x + 1$ is a one-to-one function. $x + 1 \neq y + 1$ when $x \neq y$.
- **EXAMPLE 7** If there are exactly 120 one to one functions possible from A to B then Which of the following is not true .(I)|A|=5,|B|=5 (II) |A|=4,|B|=5
- (III)|A|=3,|B|=6 (IV)|A|=5,|B|=4 (V)|A|=3,|B|=10
- *Solution:* (I) $^5P_5=\frac{5!}{(5-5)!}=120$ True (II) $^5P_4=120$ True (III) $^6P_3=120$
- (IV) |A|>|B| not one to one (V) $^{10}P_3=720$ not true
- **DEFINITION 6** A function *f* from *A* to *B* is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function *f* is called *surjective* if it is onto.
- Onto functions A→B possible if |B|≤ |**A**|
- If |A|=|B| then every one to one function from A →B is also onto and vice versa.
- If |A|=|B|=n then number of onto functions from A→B is n!.
- If |A|=m and |B|=n (m>n) then numbers of onto functions possible from A →B is
- $n^m - nc_1(n-1)^m + nc_2(n-2)^m - nc_3(n-3)^m + \ldots..(-1)^n\, nc_{n-1}(1)^m$
- **EXAMPLE 8** |A|= 6,|B|=3 then how many onto functions are possible A →B
- *Solution:* m=6 n=3 ,Numbers of onto functions possible
- $3^6 - 3c_1(3-1)^6 + 3c_2(3-2)^6 = 729-3*64+3=732-192=540$

- **EXAMPLE 9** |A|=n and B=2 (n≥2) then number of onto functions possible from A →B is ___ .
- *Solution:* $2^n - 2c_1(2-1)^n = 2^n - 2c_1(2-1)^n = 2^n - 2$
- **EXAMPLE 10** How many ways we can assign 5 employes to 4 projects so that every employ is assigned to only one project and every project is assigned by at least one employe.
- *Solution:* Number of ways possible $4^5 - 4c_1(4-1)^5 + 4c_2(4-2)^5 - 4c_3(4-3)^5$
- =1024-4*243+6*32-4
- =1024-972+192-4=1216-976=240



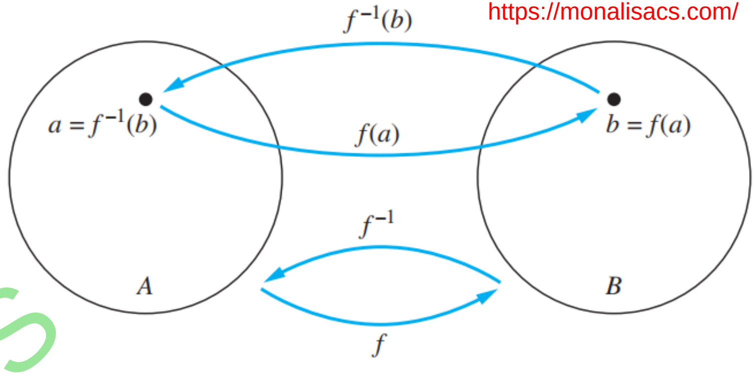
- **EXAMPLE 11** Let *f* be the function from {*a*, *b*, *c*, *d*} to {1, 2, 3} defined by *f* (*a*) = 3, *f* (*b*) = 2, *f* (*c*) = 1, and *f* (*d*) = 3. Is *f* an onto function?
- *Solution:* Because all three elements of the codomain are images of elements in the domain, we see that *f* is onto.
- **EXAMPLE 12** Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?
- *Solution:* The function *f* is not onto because there is no integer *x* with $x^2 = -1$, for instance.
- **EXAMPLE 13** Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?
- *Solution:* This function is onto , because for every integer *y* there is an integer *x* such that $f(x) = y$ if $x + 1 = y$.



- **DEFINITION 7** The function *f* is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto. We also say that such a function is *bijective*.
- Bijection functions A→B possible if |A|= |B|
- If |A|=|B|=n then number of Bijection possible =n!
- **EXAMPLE 14** Let *f* be the function from {*a, b, c, d*} to {1, 2, 3, 4} with *f (a)* = 4, *f (b)* = 2, *f (c)* = 1, and *f (d)* = 3. Is *f* a bijection?
- *Solution:* The function *f* is one-to-one and onto. Hence, *f* is a bijection.
- Suppose that *f* is a function from a set *A* to itself. If *A* is finite, then *f* is one-to-one if and only if it is onto.

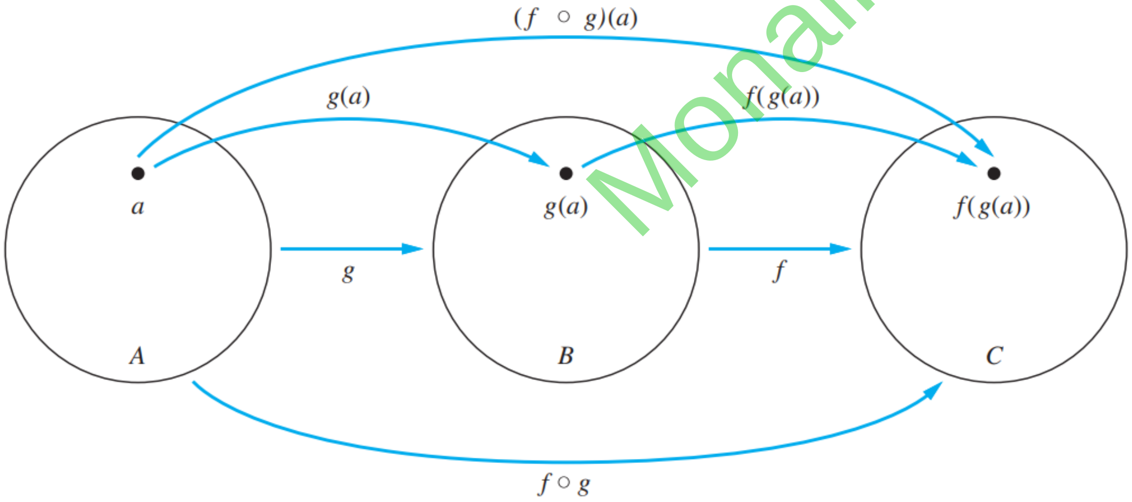
- Let A be a set. The *identity function* on A is the function $\iota_A : A \to A$, where $\iota_A(x) = x$ for all $x \in A$.
- The identity function ι_A is the function that assigns each element to itself.
- The function ι_A is one-to-one and onto, so it is a bijection.
- **Inverse Functions and Compositions of Functions**
- **DEFINITION 8** Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.
- Be sure not to confuse the function f^{-1} with the function $1/f$.
- If a function f is not a one-to-one correspondence, we cannot define an inverse function of f .
- If f is not one-to-one, some element b in the codomain is the image of more than one element in the domain.
- If f is not onto, for some element b in the codomain, no element a in the domain exists for which $f(a) = b$.
- A one-to-one correspondence is called **invertible** because we can define an inverse of this function.
- A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

- **The Function f^{-1} Is the Inverse of Function f.**



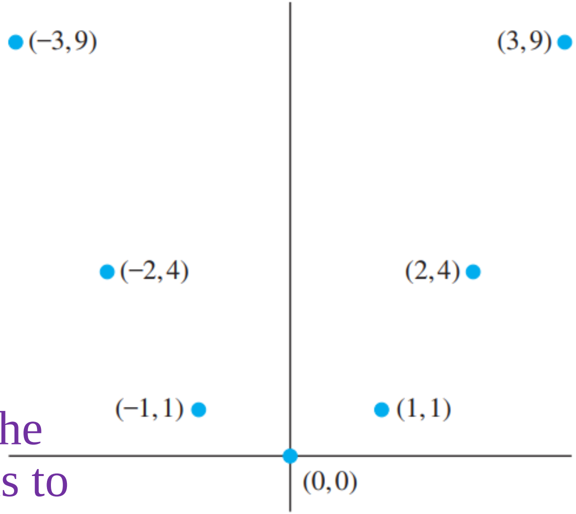
- **EXAMPLE 15** Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?
- *Solution:* The function f is invertible because it is a one-to-one correspondence.
- The inverse function f^{-1}, $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.
- **EXAMPLE 16** Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?
- *Solution:* The function f has an inverse because it is a one-to-one correspondence,
- So that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of $\mathbf{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$.
- **EXAMPLE 17** Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ with $f(x) = x^2$. Is f invertible?
- *Solution:* Because $f(-2) = f(2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible.

- **DEFINITION 9** Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.
- To find $(f \circ g)(a)$ we first apply the function g to a to obtain $g(a)$ and then we apply the function f to the result $g(a)$ to obtain $(f \circ g)(a) = f(g(a))$.
- The composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.
- The commutative law does not hold for the composition of functions, $(f \circ g) \neq (g \circ f)$
- **The Composition of the Functions *f* and g.**



- **EXAMPLE 18** Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?
- *Solution:* $(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$,
- $(f \circ g)(c) = f(g(c)) = f(a) = 3$.
- $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.
- **EXAMPLE 19** Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?
- *Solution:* $(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$
- And $(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$.
- When the composition of a function and its inverse is formed, in either order, an identity function is obtained.
- When $f(a) = b$, and $f^{-1}(b) = a$. Hence, $(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$, and $(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b$.
- Consequently $f^{-1} \circ f = \iota_A$ and $f \circ f^{-1} = \iota_B$, where ι_A and ι_B are the identity functions on the sets A and B, $(f^{-1})^{-1} = f$.

- **The Graphs of Functions**
- **DEFINITION 10** Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.
- **EXAMPLE 20** Display the graph of the function $f(x) = x^2$ from the set of integers to the set of integers.
- *Solution:* $(x, f(x)) = (x, x^2)$, The Graph of $f(x) = x^2$ from Z to Z.



- **DEFINITION 11** The *floor function* assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The *ceiling function* assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.
- The floor function is often also called the *greatest integer function*. It is denoted by $[x]$.
- These are some values of the floor and ceiling functions:
- $\left\lfloor \frac{1}{2} \right\rfloor = 0$, $\left\lceil \frac{1}{2} \right\rceil = 1$, $\left\lfloor -\frac{1}{2} \right\rfloor = -1$, $\left\lceil -\frac{1}{2} \right\rceil = 0$, $\lfloor 3.1 \rfloor = 3$,$\lceil 3.1 \rceil = 4$, $\lfloor 7 \rfloor = 7$,$\lceil 7 \rceil = 7$
- **Useful Properties of the Floor and Ceiling Functions.**
- (n is an integer, $\boldsymbol{x}$ is a real number)
- (1a) $\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
- (1b) $\lceil x \rceil = n$ if and only if $n - 1 < x \le n$

- (2) $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
- (3a) $\lfloor -x \rfloor = -\lceil x \rceil$ (3b) $\lceil -x \rceil = -\lfloor x \rfloor$
- (4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ (4b) $\lceil x + n \rceil = \lceil x \rceil + n$
- **Factorial function** $f : \mathbf{N} \to \mathbf{Z}^+$, denoted by $f(n) = n!$.
- $f(n) = 1 \cdot 2 \cdots (n - 1) \cdot n$ [and $f(0) = 0! = 1$].
- **EXAMPLE 21** We have $f(1) = 1! = 1$, $f(2) = 2! = 1 \cdot 2 = 2$,
- $f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$,
- **Partial Functions**
- **DEFINITION 12** A *partial function* f from a set A to a set B is an assignment to each element a in a subset of A, called the *domain of definition* of f, of a unique element b in B. The sets A and B are called the *domain* and *codomain* of f, respectively. We say that f is *undefined* for elements in A that are not in the domain of definition of f. When the domain of definition of f equals A, we say that f is a *total function.*
- A program may not produce a correct value because evaluating the function may lead to an infinite loop or an overflow.
- **EXAMPLE 22** The function $f : \mathbf{Z} \to \mathbf{R}$ where $f(n) = \sqrt{n}$ is a partial function from $\mathbf{Z}$ to $\mathbf{R}$ where the domain of definition is the set of nonnegative integers.

2.4 Sequences and Summations

- **Sequences**
- A sequence is a discrete structure used to represent an ordered list. For example, 1, 2, 3, 5, 8 is a sequence with five terms and 1, 3, 9, 27, 81 , . . . , $3n$, . . . is an infinite sequence.
- **DEFINITION 1** A *sequence* is a function from a subset of the set of integers to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a *term* of the sequence.
- **EXAMPLE 1** Consider the sequence $\{a_n\}$, where $a_n = \frac{1}{n}$. The list of the terms of this sequence, beginning with a_1, namely, $a_1, a_2, a_3, a_4, \ldots$, starts with $1, \frac{1}{2}, \frac{1}{3}, \ldots$
- **DEFINITION 2** A *geometric progression* is a sequence of the form $a, ar, ar^2, \ldots, ar^n, \ldots$ where the *initial term* a and the *common ratio* r are real numbers.
- A geometric progression is a exponential function $f(x) = ar^x$.
- To find the sum of finite (n) terms of a GP,
- $S_n = a(r^n - 1) / (r - 1)$
- [OR] $S_n = a(1 - r^n) / (1 - r)$, if $r \neq 1$.
- $S_n = an$, if $r = 1$.
- To find the sum of infinite terms of a GP,

  $S = a / (1 - r)$, if $|r| < 1$

- **DEFINITION 3** An *arithmetic progression* is a sequence of the form $a, a + d, a + 2d, \ldots, a + nd$ .where the *initial term a* and the *common difference d* are real numbers.
- An arithmetic progression is a linear function $f(x) = dx + a$.
- Sum = n/2 × [2a + (n-1)d]
- If a_n is known: $S_n = n/2 \times [a_1 + a_n]$
- These finite sequences are also called **strings**.
- This string is also denoted by $a_1 a_2 \ldots a_n$.
- The **length** of a string is the number of terms in this string.
- The **empty string**(λ), is the string that has no terms. The empty string has length zero.
- **DEFINITION 4** A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- **EXAMPLE 2** Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$,and suppose that $a_0 = 2$.
- What are a_1, a_2, and a_3?
- *Solution:* $a_1 = a_0 + 3 = 2 + 3 = 5$. $a_2 = 5 + 3 = 8$ and $a_3 = 8 + 3 = 11$.

- **EXAMPLE 3** Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n =$ 2, 3, 4, . . . , and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?
- *Solution:* $a_2 = a_1 - a_0 = 5 - 3 = 2$ and $a_3 = a_2 - a_1 = 2 - 5 = -3$.
- The **initial conditions** for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect.
- **DEFINITION 5** The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots,$ is defined by the initial conditions $f_0=0$, $f_1=1$, and the recurrence relation $f_n=f_{n-1}+ f_{n-2}$ for $n =2,3,4,\ldots$.
- **EXAMPLE 4** Find the Fibonacci numbers $f_2,f_3,f_4,f_5,$ and f_6 the initial conditions $f_0=0$ and $f_1=1$
- *Solution:* $f_2 = f_1 + f_0 = 1 + 0 = 1$, $f_3 = f_2 + f_1 = 1 + 1 = 2$,
- $f_4 = f_3 + f_2 = 2 + 1 = 3$, $f_5 = f_4 + f_3 = 3 + 2 = 5$, $f_6 = f_5 + f_4 = 5 + 3 = 8$.
- We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a **closed formula**, for the terms of the sequence.
- **EXAMPLE 5** Solve the recurrence relation and initial condition $a_n=a_{n-1}+3$ for $n=1,2,3,\ldots,$ and suppose that $a_1 = 2$.
- *Solution:* Starting with the initial condition $a_1 = 2$, and working upward until we reach a_n to deduce a closed formula for the sequence.
- $a_2 = 2 + 3$ $a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$
- $a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$ ..
- $a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$.

- We can also successively apply the recurrence relation starting with the term a_n and working downward until we reach the initial condition $a_1 = 2$ to deduce this same formula.
- $a_n = a_{n-1} + 3 \quad = (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2$
- $= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3$ .....
- $= a_2 + 3(n - 2) = (a_1 + 3) + 3(n - 2) = 2 + 3(n - 1).$
- The technique used in Example is called **iteration**. We have iterated, or repeatedly used,the recurrence relation.
- The first approach is called **forward substitution** – we found successive terms beginning with the initial condition and ending with a_n.
- The second approach is called **backward substitution**, because we began with a_n and iterated to express it in terms of falling terms of the sequence until we found it in terms of a_1.
- **EXAMPLE 6** Find formulae for the sequences with the following first five terms: (a) 1, 1/2, 1/4, 1/8, 1/16 (b) 1, 3, 5, 7, 9 (c) 1, -1, 1, -1, 1.
- *Solution:* (a) The sequence with $a_n = 1/2^n, n = 0, 1, 2, \ldots$.
- This is a geometric progression with $a = 1$ and $r = 1/2$.
- (b) Each term is obtained by adding 2 to the previous term. The sequence with $a_n = 2n + 1$, $n = 0, 1, 2, \ldots$.This is an arithmetic progression with a = 1 and d = 2.
- (c) The terms alternate between 1 and -1. The sequence with $a_n = (-1)^n$, $n = 0,1,2\ldots$.This sequence is a geometric progression with a = 1 and r = -1 .

- **EXAMPLE 7** Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.
- *Solution:* this sequence are generated by a formula involving 3^n.
- Comparing these terms with the corresponding terms of the sequence $\{3^n\}$, we notice that the nth term is 2 less than the corresponding power of 3.
- $a_n = 3^n - 2$ for $1 \le n \le 10$ and conjecture that this formula holds for all n.

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

- **Summations**
- $a_m + a_{m+1}, \ldots, + a_n$ We use the notation $\sum_{J=m}^{n} a_j$, $\Sigma_{m \le j \le n} a_j$
- Here, the variable *j* is called the **index of summation**.
- Here, the index of summation runs through all integers starting with its **lower limit** *m* and ending with its **upper limit** *n*.
- A large uppercase Greek letter sigma, *Σ*, is used to denote summation.
- $\sum_{J=1}^{n} (a x_j + b y_j) = a \sum_{j=1}^{n} x_j + \mathrm{b} \sum_{j=1}^{n} y_j$
- **EXAMPLE 8** Use summation notation to express the sum of the first 100 terms of the sequence $\{a_j\}$, where $a_j = 1/j$ for $j = 1, 2, 3, \ldots$.
- *Solution:* The lower limit for the index of summation is 1, upper limit is 100. $\sum_{J=1}^{100} {}^{1}/_{J}$.
- **EXAMPLE 9** What is the value of $\sum_{j=1}^{5} j^2$?
- *Solution:* $\sum_{J=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$.
- **EXAMPLE 10** $\sum_{i=1}^{4} * \sum_{j=1}^{3} ij$
- *Solution:* first expand the inner summation and then continue by computing the outer summation:

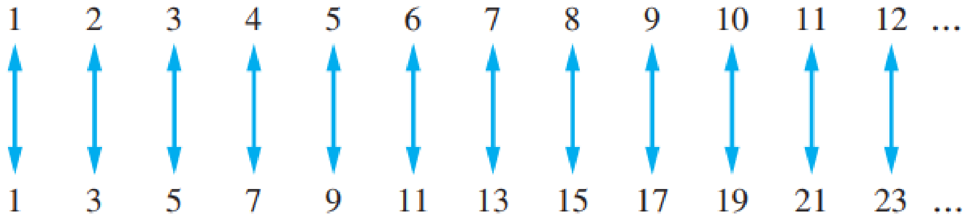
- $\sum_{i=1}^{4} * \sum_{j=1}^{3} iJ = \sum_{i=1}^{4}(i + 2i + 3i)$
- $\sum_{i=1}^{4} 6i = 6 \sum_{i=1}^{4} i = 6(1+2+3+4) = 6*10 = 60$
- **THEOREM 1** If a and r are real numbers and $r \neq 0$, then
- $$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$
- **EXAMPLE 11** What is the value of $\Sigma_{s\epsilon\{0,2,4\}} s$?
- *Solution:* $\Sigma_{s\epsilon\{0,2,4\}} s = 0+2+4 = 6$
- **EXAMPLE 12** Find $\sum_{k=50}^{100} k^2$
- *Solution:* $\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$
- $= \frac{100\times101\times201}{6} - \frac{49\times50\times99}{6}$
- $=338350-40425=297925$

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

2.5 Cardinality of Sets

- **DEFINITION 1** The sets A and B have the same *cardinality* if and only if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write $|A| = |B|$
- **DEFINITION 2** If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.
- **Countable Sets**
- **DEFINITION 3** A set that is either finite or has the same cardinality as the set of positive integers is called *countable.* A set that is not countable is called *uncountable.*
- **EXAMPLE 1** Show that the set of odd positive integers is a countable set.
- *Solution:* Consider $f(n) = 2n - 1$ from $\mathbf{Z}^+$ to the set of odd positive integers.
- **FIGURE 1** **A One-to-One Correspondence Between Z^+ and the Set of Odd Positive Integers.**



- EXAMPLES OF COUNTABLE AND UNCOUNTABLE SETS
- Set of all integers is countable.
- **EXAMPLE 2** Show that the set of positive rational numbers is countable.
- *Solution:* list the positive rational numbers as a sequence $r_1, r_2, \ldots, r_n, \ldots$.
- Every positive rational number is the quotient p/q of two positive integers.
- Arrange the positive rational numbers by listing those with denominator q = 1 in the first row , q = 2 in the second row, and so on.
- First list the positive rational numbers p/q with $p + q = 2$, followed by those with $p + q = 3$, followed by those with $p + q = 4$, and so on.
- Whenever we encounter a number p/q that is already listed, we do not list it again.
- For example,when we come to 2/2 = 1 we do not list it because we have already listed 1/1= 1.
- Because all positive rational numbers are listed once, as the reader can verify, the set of positive rational numbers is countable.

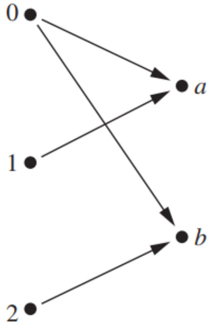
An Uncountable Set

- **EXAMPLE 3** Show that the set of real numbers is an uncountable set.
- *Solution:* To show that the set of real numbers is uncountable, we suppose that the set of real numbers is countable and arrive at a contradiction.
- Then, the subset of all real numbers that fall between 0 and 1 would also be countable (because any subset of a countable set is also countable).
- The real numbers between 0 and 1 can be listed in some order, say, $r1, r2, r3, \ldots$. Let the decimal representation of these real numbers be
- $r1 = 0.d11d12d13d14 \ldots$
- $r2 = 0.d21d22d23d24 \ldots$
- $r3 = 0.d31d32d33d34 \ldots$
- Example $r1$ = 0.23794102…,$r2$ = 0.44590138…,$r3$ =0.09118764 ... , and so on.
- Therefore, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable.
- Any set with an uncountable subset is uncountable.
- Hence, the set of real numbers is uncountable.

- **THEOREM 1** If A and B are countable sets, then $A \cup B$ is also countable
- **DEFINITION 4** We say that a function is **computable** if there is a computer program in some programming language that finds the values of this function. If a function is not computable we say it is **uncomputable** .
- Every infinite set S contains a countable subset.
- Every subset of countable set is countable.
- Power set of countable set is uncountable.
- Set of all integers is countable
- Set of positive rational numbers is countable
- The set R is uncountable.
- The set Z^2 is countable.
- Q is countable.
- The set of infinite sequences is uncountable.
- The set of finite sequences is countable.

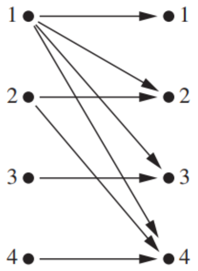
2.6 Relations and Their Properties

- **DEFINITION 1** Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$
- A binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.
- We use the notation $a\ R\ b$ to denote that $(a, b) \in R$ and a *not* $R\ b$ to denote that $(a, b) \notin R$.
- a is said to be **related to** b by R.
- **EXAMPLE 1** Let A be the set of cities in the U.S.A., and let B be the set of the states in the U.S.A. Define the relation R by specifying that (a, b) belongs to R if a city with name a is in the state b. For instance, (Naperville, Illinois), (Dells,Wisconsin), (Chicago, Illinois), (Middletown, New Jersey), are in R.
- **EXAMPLE 2** Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. Relations can be represented graphically, using arrows to represent ordered pairs. Another way to represent this relation is to use a table.



R	a	b
0	×	×
1	×	
2		×

- **Functions as Relations** The graph of f is the set of ordered pairs (a, b) such that $b = f(a)$. Because the graph of f is a subset of $A \times B$, it is a relation from A to B.
- **Relations on a Set**
- **DEFINITION 2** A *relation on a set A* is a relation from A to A.
- In other words, a relation on a set A is a subset of $A \times A$.
- **EXAMPLE 3** Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

- *Solution:* $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
- **EXAMPLE 4** How many relations are there on a set with n elements?
- *Solution:* A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements.
- For example, there are $2^{3^2} = 2^9 = 512$ relations on the set $\{a, b, c\}$.
- If $|A|=m$ and $|B|=n$ then number of relations possible from A to $B=2^{m*n}$
- If $|A|=n$ then number of relations possible A to $A=2^{n^2}$
- ***Inverse of relation*** Let R be a relation from A to B .R^{-1} ={(b,a) , (a,b)∈ R} is called inversal of R and R^{-1} is a relation from B to A.
- ***Complement of a Relation*** $\bar{R}$=($A \times$ B)-R Complement of R

- **Diagonal Relation** A relation R on a set A is called diagonal relation on A if R={(a,a)|∀a∈A}
- If A={1,2,3} , Δ_A={(1,1),(2,2),(3,3)}
- **Properties of Relations**
- **DEFINITION 3** A relation *R* on a set *A* is called *reflexive* if *(a, a)* ∈ *R* , ∀*a* ∈ *A*
- The diagonal relation on A is the smallest reflexive relation on A.
- Any superset of diagonal relation is also reflexive .
- Let A={a,b,c}
- R_1= {(a,a),(b,b),(c,c)} smallest reflexive relation
- R_2= {(a,a),(a,b),(b,b),(c,a),(c,c)}
- R_3=A ×A Largest reflexive relation
- **Number of Reflexive Relations**
- Consider a set A with n elements , Say A ={1, 2, ……. n−1, n }.|A ×A|=n^2
- Out of n^2 elements n elements are compulsory for relation to be reflexive.
- i.e (1, 1) (2, 2) (3, 3) …. (n, n)
- Remaining $n^2 - n$ elements, we have choice of filling i.e either they are present or absent.
- Hence, Total number of reflexive relation are $2^{n^2 - n} = 2^{n(n-1)}$.
- Number of relations not reflexive = $2^{n^2} - 2^{n^2 - n}$

- **EXAMPLE 5** Consider the following relations on {1, 2, 3, 4}:
- $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,
- $R_2 = \{(1, 1), (1, 2), (2, 1)\}$,
- $R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,
- $R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,
- $R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,
- $R_6 = \{(3, 4)\}$. Which of these relations are reflexive?
- *Solution:* The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (a, a), namely, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$.
- R_1, R_2, R_4, and R_6 are not reflexive because $(3, 3)$ is not in any of these relations.
- **EXAMPLE 6** Consider these relations on the set of integers:
- $R_1 = \{(a, b) \mid a \leq b\}$, $R_2 = \{(a, b) \mid a > b\}$,
- $R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$, $R_4 = \{(a, b) \mid a = b\}$,
- $R_5 = \{(a, b) \mid a = b + 1\}$, $R_6 = \{(a, b) \mid a + b \leq 3\}$.
- Which of these relations are reflexive?
- *Solution:* R_1 (because $a \leq a$ for every integer a), R_3, and R_4.

- **DEFINITION 4** A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.
- The relation R on the set A is symmetric if $\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$. Similarly, the relation R on the set A is antisymmetric if $\forall a \forall b(((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b))$.
- **EXAMPLE 7** if A={a,b,c} then
- R_1={ }smallest symmetric / antisymmetric relation on A
- R_2={(b,b)(c,c)} symmetric / antisymmetric relation
- R_3={(a,b),(b,a),(b,c)(c,b)} symmetric but not antisymmetric relation
- R_4=A×A largest symmetric relation on A.
- A relation is symmetric if and only if *a* is related to *b* implies that *b* is related to *a*.
- A relation is antisymmetric if and only if there are no pairs of distinct elements *a* and *b* with *a* related to *b* and *b* related to *a*.
- The terms *symmetric* and *antisymmetric* are not opposites,because a relation can have both of these properties or may lack both of them .
- If A is a set with n elements then number of symmetric relations possible $=2^{\frac{n(n+1)}{2}}$
- Numbers of Symmetric relations possible with diagonal pair=2^n

- Numbers of Symmetric relation possible with non diagonal pair= $2^{\frac{n(n-1)}{2}}$
- Total Symmetric relations possible = $2^n * 2^{\frac{n(n-1)}{2}} = 2^{\frac{2n+n(n-1)}{2}} = 2^{\frac{2n+n^2-n}{2}} = 2^{\frac{n^2+n}{2}}$
- **EXAMPLE 8** Which of the relations from Example 5 are symmetric and which are antisymmetric?
- *Solution:* $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,
- $R_2 = \{(1, 1), (1, 2), (2, 1)\}$,
- $R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,
- The relations R_2 and R_3 are symmetric, because in each case *(b, a)* belongs to the relation whenever *(a, b)* does.
- $R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,
- $R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,
- $R_6 = \{(3, 4)\}$.
- R_4, R_5, and R_6 are all antisymmetric. For each of these relations there is no pair of elements *a* and *b* with $a \neq b$ such that both *(a, b)* and *(b, a)* belong to the relation.
- **EXAMPLE 9** Which of the relations from Example 6 are symmetric and which are antisymmetric?

- $R_1 = \{(a, b) \mid a \leq b\}$, $R_2 = \{(a, b) \mid a > b\}$, $R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,
- $R_4 = \{(a, b) \mid a = b\}$, $R_5 = \{(a, b) \mid a = b + 1\}$, $R_6 = \{(a, b) \mid a + b \leq 3\}$.
- *Solution:* The relations R_3, R_4, and R_6 are symmetric.
- R_3 is symmetric, for if $a = b$ or $a = -b$,then $b = a$ or $b = -a$.
- R_4 is symmetric because $a = b$ implies that $b = a$.
- R_6 is symmetric because $a + b \leq 3$ implies that $b + a \leq 3$.
- The relations R_1, R_2, R_4, and R_5 are antisymmetric.
- R_1 is antisymmetric because the inequalities $a \leq b$ and $b \leq a$ imply that $a = b$.
- R_2 is antisymmetric because it is impossible that $a > b$ and $b > a$.
- R_4 is antisymmetric, because two elements are related if they are equal.
- R_5 is antisymmetric because it is impossible that $a = b + 1$ and $b = a + 1$.
- **EXAMPLE 10** Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?
- *Solution:* Not symmetric because 1 devides 2, but 2 not devides 1.
- Antisymmetric, if a and b are positive integers with $a|b$ and $b|a$, then $a = b$

- **DEFINITION 5** A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.
- $\forall a \forall b \forall c(((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$.
- if A={a,b,c} then
- R_1={ }smallest *transitive* relation on A
- R_2={(a,a),(b,b),(c,c)}
- R_3={(a,b),(a,c)}
- R_4={(a,b),(b,c),(a,c)}
- R_5=A×A largest *transitive* relation on A.
- ≤,≥ ,<,> =,/,⊆ are transitive.
- **EXAMPLE 11** Which of the relations in Example 5 are transitive?
- *Solution:* $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,
- R_1 is not transitive because $(3, 4)$ and $(4, 1)$ belong to $R1$, but $(3, 1)$ does not.
- $R_2 = \{(1, 1), (1, 2), (2, 1)\}$,
- R_2 is not transitive because $(2, 1)$ and $(1, 2)$ belong to R_2, but $(2, 2)$ does not.
- $R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,
- R_3 is not transitive because $(4, 1)$ and $(1, 2)$ belong to R_3, but $(4, 2)$ does not.

- $R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,
- R_4 is transitive, because $(3, 2)$ and $(2, 1)$, $(4, 2)$ and $(2, 1)$, $(4, 3)$ and $(3, 1)$, and $(4, 3)$ and $(3, 2)$ are the only such sets of pairs, and $(3, 1)$, $(4, 1)$, and $(4, 2)$ belong to R_4.
- $R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,
- $R_6 = \{(3, 4)\}$.
- R_4, R_5, and R_6 are transitive.
- **EXAMPLE 12** Which of the relations are transitive?
- $R_1 = \{(a, b) \mid a \leq b\}$, $R_2 = \{(a, b) \mid a > b\}$, $R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,
- $R_4 = \{(a, b) \mid a = b\}$, $R_5 = \{(a, b) \mid a = b + 1\}$, $R_6 = \{(a, b) \mid a + b \leq 3\}$.
- *Solution:* The relations R_1, R_2, R_3, and R_4 are transitive.
- R_1 is transitive because $a \leq b$ and $b \leq c$ imply that $a \leq c$.
- R_2 is transitive because $a > b$ and $b > c$ imply that $a > c$.
- R_3 is transitive because $a = \pm b$ and $b = \pm c$ imply that $a = \pm c$.
- R_4 is clearly transitive , a=b and b=c imply that a=c.
- R_5 is not transitive because $(2, 1)$ and $(1, 0)$ belong to R_5, but $(2, 0)$ does not.
- R_6 is not transitive because $(2, 1)$ and $(1, 2)$ belong to R_6, but $(2, 2)$ does not.

- **EXAMPLE 13** Is the "divides" relation on the set of positive integers transitive?
- *Solution:* Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c.
- It follows that this relation is transitive.
- **Combining Relations**
- Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.
- **EXAMPLE 14** Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain
- $R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$,
- $R_1 \cap R_2 = \{(1, 1)\}$,
- $R_1 - R_2 = \{(2, 2), (3, 3)\}$,
- $R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}$.
- **DEFINITION 6** Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

- Computing the composite of two relations requires that we find elements that are the second element of ordered pairs in the first relation and the first element of ordered pairs in the second relation.
- **EXAMPLE 15** What is the composite of the relations R and S, where R is the relation from {1, 2, 3} to {1, 2, 3, 4} with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from {1, 2, 3, 4} to {0, 1, 2} with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?
- *Solution :* $S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.
- **DEFINITION 7** Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.
- The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on
- **EXAMPLE 16** Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers R^n, $n = 2, 3, 4, \ldots$.
- *Solution:* Because $R^2 = R \circ R$, we find that $R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.
- $R^3 = R^2 \circ R$, $R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.
- R^4 is the same as R^3, so $R^4 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.
- It also follows that $R^n = R^3$ for $n = 5, 6, 7, \ldots$.
- **THEOREM 1** The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

2.7 *n*-ary Relations and Their Applications

- **DEFINITION 1** Let $A_1, A_2, \ldots, A_n$ be sets. An *n-ary relation* on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the *domains* of the relation, and n is called its *degree*.
- **EXAMPLE 1** Let R be the relation on $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$ consisting of triples (a, b, c), where a, b, and c are integers with $a < b < c$. Then $(1, 2, 3) \in R$, but $(2, 4, 3) \notin R$. The degree of this relation is 3. Its domains are all equal to the set of natural numbers.
- **Databases and Relations**
- We can represent databases in **relational data model**, based on the concept of a relation.
- A database consists of **records**, which are n-tuples, made up of **fields**.
- The fields are the entries of the n-tuples.
- For instance, a database of student records may be made up of fields containing the name, student number, major, and grade point average of the student.

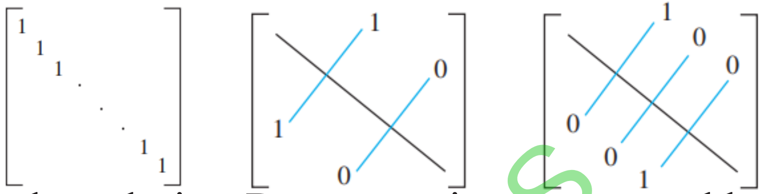
TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

2.8 Representing Relations

- **Representing Relations Using Matrices**
  A relation between finite sets can be represented using a zero–one matrix.
- $m_{ij}=1$ if $(a_i,b_j) \in R$, 0 if (a_i,b_j) not $\in R$
- The zero–one matrix representing *R* has a 1 as its *(i, j)* entry when a_i is related to b_j, and a 0 in this position if a_i is not related to b_j.
- **EXAMPLE 1** Suppose that *A* = {1, 2, 3} and *B* = {1, 2}. Let *R* be the relation from *A* to *B* containing *(a, b)* if *a* ∈ *A*, *b* ∈ *B*, and *a* > *b*.
- *Solution:* Because *R* = {(2, 1), (3, 1), (3, 2)}, the matrix for *R* is $\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$
- The matrix of a relation on a set, which is a square matrix, can be used to determine whether the relation has certain properties.
- A relation *R* on *A* is reflexive if *(a, a)* ∈ *R* whenever *a* ∈ *A*. *R* is reflexive if all the elements on the main diagonal of $\mathbf{M}_R$ are equal to 1.
- The relation *R* is symmetric if *(a, b)* ∈ *R* implies that *(b, a)* ∈ *R*.
- *R* is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers *i* and *j*.
- *R* is symmetric if and only if $\mathbf{M}_R = (\mathbf{M}_R)^t$
- The relation *R* is antisymmetric if and only if *(a, b)* ∈ *R* and *(b, a)* ∈ *R* imply that *a* = *b*.

- The matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.
- **The Zero–One Matrices for Reflexive Symmetric Antisymmetric**



$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- **EXAMPLE 2** Suppose that the relation R on a set is represented by the matrix
- Is R reflexive, symmetric, and/or antisymmetric?
- *Solution:* Because all the diagonal elements of this matrix are equal to 1, R is reflexive.
- Because $\mathbf{M}_R$ is symmetric, it follows that R is symmetric. R is not antisymmetric.
- The Boolean operations join and meet can be used to find the matrices representing the union and the intersection of two relations. $\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2}$ and $\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}$.
- **EXAMPLE 3** Suppose that the relations R_1 and R_2 on a set A are represented by the matrices
- What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

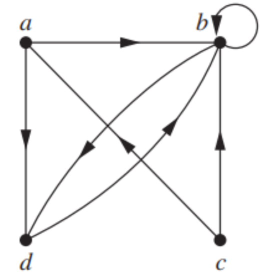
$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

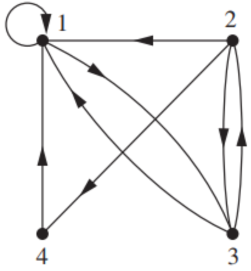
$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Representing Relations Using Digraphs

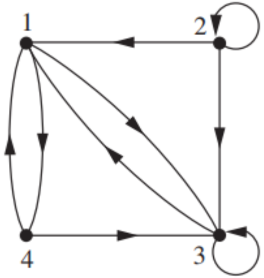
- **DEFINITION 1** A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the *terminal vertex* of this edge.
- An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a **loop**.
- **EXAMPLE 4** The directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b)



- **EXAMPLE 5** The directed graph of the relation $R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$ on the set $\{1, 2, 3, 4\}$

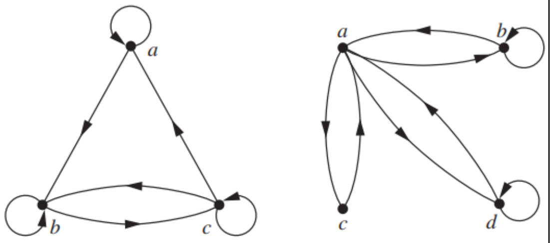


- **EXAMPLE 6** What are the ordered pairs in the relation R represented by the directed graph shown in Figure



- *Solution:* The ordered pairs (x, y) in the relation are $R = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}$.
- The directed graph representing a relation can be used to determine whether the relation has various properties.
- A relation is reflexive if and only if there is a loop at every vertex of the directed graph, so that every ordered pair of the form (x, x) occurs in the relation.

- A relation is symmetric if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction, so that *(y, x)* is in the relation whenever *(x, y)* is in the relation.
- Similarly, a relation is antisymmetric if and only if there are never two edges in opposite directions between distinct vertices.
- Finally, a relation is transitive if and only if whenever there is an edge from a vertex x to a vertex y and an edge from a vertex y to a vertex z, there is an edge from x to z .
- **EXAMPLE 10** Determine whether the relations for the directed graphs shown in Figure are reflexive, symmetric, antisymmetric, and/or transitive.



(a) Directed graph of *R* (b) Directed graph of *S*

- *Solution:* Because there are loops at every vertex of the directed graph of R, it is reflexive.
- R is neither symmetric nor antisymmetric because there is an edge from a to b but not one from b to a.
- Finally, R is not transitive because there is an edge from a to b and an edge from b to c, but no edge from a to c.
- Because loops are not present at all the vertices of the directed graph of S, this relation is not reflexive.

- It is symmetric and not antisymmetric, because every edge between distinct vertices is accompanied by an edge in the opposite direction.
- It is also not hard to see from the directed graph that S is not transitive, because (c, a) and (a, b) belong to S, but (c, b) does not belong to S.

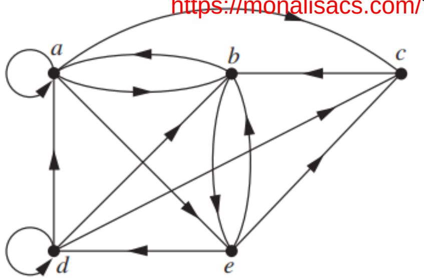
2.9 Closures of Relations

- **Introduction :** let R be a relation on a set A. R may or may not have some property **P**, such as reflexivity, symmetry, or transitivity. If there is a relation S with property **P** containing R such that S is a subset of every relation with property **P** containing R, then S is called the **closure** of R with respect to P.
- **Closures :** The relation $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on the set $A = \{1, 2, 3\}$ is not reflexive.
- How can we produce a reflexive relation containing R that is as small as possible? This can be done by adding $(2, 2)$ and $(3, 3)$ to R, because these are the only pairs of the form (a, a) that are not in R.
- Because this relation contains R, is reflexive, and is contained within every reflexive relation that contains R, it is called the **reflexive closure** of R.
- The reflexive closure of R can be formed by adding to R all pairs of the form (a, a) with $a \in A$, not already in R. The reflexive closure of $R = R \cup \Delta$, where $\Delta = \{(a,a) | a \in A\}$ is the **diagonal relation** on A.

- **EXAMPLE 1** What is the reflexive closure of the relation $R = \{(a,b) | a<b\}$ on the set of integers?
- *Solution:* The reflexive closure of R is $R \cup \Delta = \{(a, b) | a < b\} \cup \{(a, a) | a \in \mathbf{Z}\} = \{(a, b) \mid a \leq b\}$.
- The relation $\{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)\}$ on $\{1, 2, 3\}$ is not symmetric.
- How can we produce a symmetric relation that is as small as possible and contains R?
- To do this,we need only add $(2, 1)$ and $(1, 3)$, because these are the only pairs of the form (b, a) with $(a, b) \in R$ that are not in R.
- This new relation is symmetric and contains R.
- This new relation is called the **symmetric closure** of R.
- The symmetric closure of a relation can be constructed by taking the union of a relation with its inverse that is, $R \cup R^{-1}$ is the symmetric closure of R, where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.
- **EXAMPLE 2** What is the symmetric closure of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers?
- *Solution:* The symmetric closure of R is the relation $R \cup R^{-1} = \{(a, b) | a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \neq b\}$.
- Suppose that a relation R is not transitive. How can we produce a transitive relation that contains R such that this new relation is contained within any transitive relation.

- Consider the relation $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on the set $\{1, 2, 3, 4\}$. This relation is not transitive because it does not contain all pairs of the form (a, c) where (a, b) and (b, c) are in R.
- The pairs of this form not in R are $(1, 2)$, $(2, 3)$, $(2, 4)$, and $(3, 1)$. Adding these pairs does *not* produce a transitive relation, because the resulting relation contains $(3, 1)$ and $(1, 4)$ but does not contain $(3, 4)$.
- This shows that constructing the transitive closure of a relation is more complicated than constructing either the reflexive or symmetric closure.
- The transitive closure of a relation can be found by adding new ordered pairs that must be present and then repeating this process until no new ordered pairs are needed.
- **Paths in Directed Graphs**
- **DEFINITION 1** A *path* from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a nonnegative integer, and $x_0 = a$ and $x_n = b$,that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has *length* n. We view the empty set of edges as a path of length zero from a to a. A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

- **EXAMPLE 3** Which of the following are paths in the directed graph shown in Figure : *a, b, e, d*; *a, e, c, d, b*; *b, a, c, b, a, a, b*; *d, c*; *c, b, a*; *e, b, a, b, a, b, e*? What are the lengths of those that are paths? Which of the paths in this list are circuits?



- *Solution:* Because each of *(a, b)*, *(b, e)*, and *(e, d)* is an edge, *a, b, e, d* is a path of length three.
- Because *(c, d)* is not an edge, *a, e, c, d, b* is not a path.
- Also, *b, a, c, b, a, a, b* is a path of length six because *(b, a)*, *(a, c)*, *(c, b)*, *(b, a)*, *(a, a)*, and *(a, b)* are all edges.
- We see that *d, c* is a path of length one, because *(d, c)* is an edge.
- Also *c, b, a* is a path of length two, because *(c, b)* and *(b, a)* are edges.
- All of *(e, b)*, *(b, a)*, *(a, b)*, *(b, a)*, *(a, b)*, and *(b, e)* are edges, so *e, b, a, b, a, b, e* is a path of length six.
- The two paths *b, a, c, b, a, a, b* and *e, b, a, b, a, b, e* are circuits because they begin and end at the same vertex.
- The paths *a, b, e, d*; *c, b, a*; and *d, c* are not circuits.

- There is a **path** from a to b in R if there is a sequence of elements $a, x_1, x_2, \ldots, x_{n-1}, b$ with $(a, x_1) \in R, (x_1, x_2) \in R, \ldots,$ and $(x_{n-1}, b) \in R$.
- **Transitive Closures**
- **DEFINITION 2** Let R be a relation on a set A. The *connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.
- **THEOREM 2** The transitive closure of a relation R equals the connectivity relation R^*
- **THEOREM 3** Let $\mathbf{M}_R$ be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is $\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}$.
- Warshall's algorithm can be used for finding transitive Closures.

2.10 Equivalence Relations

- **DEFINITION 1** A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.
- **DEFINITION 2** Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.
- **EXAMPLE 1** A={a,b,c}
- R_1={(a,a)(b,b)(c,c)} smallest equivalence relation on A.
- R_2={(a,a)(b,b)(c,c)(a,b)(b,a)}
- R_2=A ×A largest equivalence relation on A.
- **EXAMPLE 2** Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. we showed that R is reflexive, symmetric, and transitive.It follows that R is an equivalence relation.
- **EXAMPLE 3** Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer.Is R an equivalence relation?
- *Solution:* Because $a - a = 0$ is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive.
- Now suppose that aRb. Then $a - b$ is an integer, so $b - a$ is also an integer. Hence, bRa. It follows that R is symmetric.

- If aRb and bRc, then $a - b$ and $b - c$ are integers. Therefore, $a - c = (a - b) + (b - c)$ is also an integer. Hence, aRc. Thus, R is transitive.
- Consequently, R is an equivalence relation.
- **Equivalence Classes**
- **DEFINITION 3** Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.
- If R is an equivalence relation on a set A, the equivalence class of the element a is $[a]_R = \{s \mid (a, s) \in R\}$.
- If $b \in [a]_R$, then b is called a **representative** of this equivalence class.
- **EXAMPLE 4** What is the equivalence class of an integer for the equivalence relation of Example 2?
- *Solution:* Because an integer is equivalent to itself and its negative in this equivalence relation, it follows that $[a] = \{-a, a\}$.
- This set contains two distinct integers unless $a = 0$.
- For instance, $[7] = \{-7, 7\}$, $[-5] = \{-5, 5\}$, and $[0] = \{0\}$.

- **EXAMPLE 5** What are the equivalence classes of 0 and 1 for congruence modulo 4?
- *Solution:* The equivalence class of 0 contains all integers a such that $a \equiv 0 \pmod 4$.
- The integers in this class are those divisible by 4.
- Hence, the equivalence class of 0 for this relation is $[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$.
- The equivalence class of 1 contains all the integers a such that $a \equiv 1 \pmod 4$.
- The integers in this class are those that have a remainder of 1 when divided by 4.
- Hence, the equivalence class of 1 for this relation is $[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$.
- The equivalence classes of the relation congruence modulo m are called the **congruence classes modulo** m.
- The congruence class of an integer a modulo m is denoted by $[a]_m$,
- so $[a]_m = \{\ldots, a - 2m, a - m, a, a + m, a + 2m, \ldots\}$.

2.11 Partial Orderings

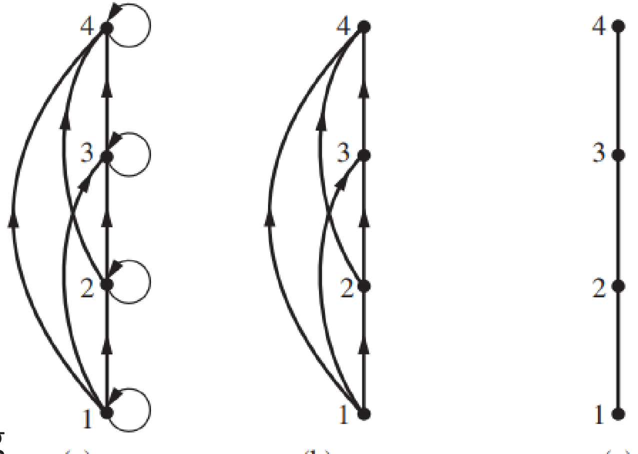
- **DEFINITION 1** A relation *R* on a set *S* is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set *S* together with a partial ordering *R* is called a *partially ordered set*, or *poset*, and is denoted by *(S, R)*. Members of *S* are called *elements* of the poset.
- **EXAMPLE 1** if A={a,b,c}
- R_1={(a,a)(b,b)(c,c)} smallest poset and only relation which is both equivalence and poset.
- R_2={(a,a)(b,b)(c,c)(a,b)(a,c)}
- **EXAMPLE 2** Show that the "greater than or equal" relation (≥) is a partial ordering on the set of integers.
- *Solution:* Because $a \geq a$ for every integer a, ≥ is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$.
- Hence, ≥ is antisymmetric. Finally, ≥ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$.
- It follows that ≥ is a partial ordering on the set of integers and $(\mathbf{Z}, \geq)$ is a poset.
- **EXAMPLE 3** The divisibility relation | is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive.
- **EXAMPLE 4** Show that the relation ⊆ is a partial ordering on the power set of a set *S*.
- *Solution:* Because $A \subseteq A$ whenever A is a subset of S, ⊆ is reflexive.
- It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$.

- Finally, ⊆ is transitive, because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$.
- Hence, ⊆ is a partial ordering on $P(S)$, and $(P(S), \subseteq)$ is a poset.
- **EXAMPLE 5** Let R be the relation on the set of people such that $xR\,y$ if x and y are people and x is older than y. Show that R is not a partial ordering.
- *Solution:* Note that R is antisymmetric because if a person x is older than a person y, then y is not older than x. That is, if $xR\,y$, then y *not* $R\,x$.
- The relation R is transitive because if person x is older than person y and y is older than person z, then x is older than z. That is, if $xR\,y$ and yRz, then $xR\,z$.
- However, R is not reflexive, because no person is older than himself or herself.
- That is, x *not* $R\,x$ for all people x. It follows that R is not a partial ordering.
- **DEFINITION 2** The elements a and b of a poset $(S, \leq)$ are called *comparable* if either $a \leq b$ or $b \leq a$. When a and b are elements of S such that neither $a \leq b$ nor $b \leq a$, a and b are called *incomparable*.
- **EXAMPLE 6** In the poset $(\mathbf{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?
- *Solution:* The integers 3 and 9 are comparable, because 3 | 9. The integers 5 and 7 are incomparable, because 5 not | 7 and 7 not | 5.
- When every two elements in the set are comparable, the relation is called a **total ordering**.

- **DEFINITION 3** If $(S, \leq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and is called a *total order* or a *linear order.* A totally ordered set is also called a *chain*.
- **EXAMPLE 7** The poset $(\mathbf{Z}, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers.
- **EXAMPLE 8** The poset $(\mathbf{Z}^+, |)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7.
- **DEFINITION 4** $(S, \leq)$ is a *well-ordered set* if it is a poset such that is a total ordering and every nonempty subset of S has a least element.
- **Lexicographic Order**
- The **lexicographic ordering** $\leq$on $A_1 \times A_2$ is defined by specifying that one pair is less than a second pair if the first entry of the first pair is less than (in A_1) the first entry of the second pair, or if the first entries are equal, but the second entry of this pair is less than (in A_2) the second entry of the second pair. In other words, (a_1, a_2) is less than (b_1, b_2), that is,
- $(a_1, a_2) \prec (b_1, b_2)$,
- either if $a_1 \prec_1 b_1$ or if both $a_1 = b_1$ and $a_2 \prec_2 b_2$.
- **EXAMPLE 9** Determine whether $(3, 5) \prec (4, 8)$, whether $(3, 8) \prec (4, 5)$, and whether $(4, 9) \prec (4, 11)$ in the poset $(\mathbf{Z} \times \mathbf{Z}, \leq)$,

- *Solution:* Because $3 < 4$, it follows that $(3, 5) \prec (4, 8)$ and that $(3, 8) \prec (4, 5)$.
- We have $(4, 9) \prec (4, 11)$, because the first entries of $(4, 9)$ and $(4, 11)$ are the same but $9 < 11$.

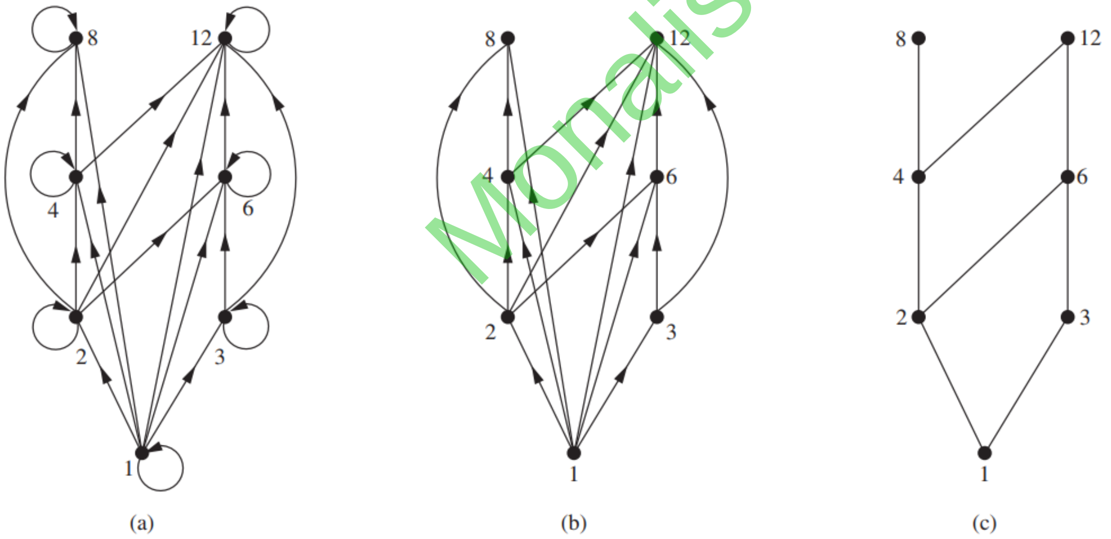
- **Hasse Diagrams**
- Consider the directed graph for the partial ordering $\{(a, b) \mid a \leq b\}$ on the set $\{1, 2, 3, 4\}$, shown in Figure (a).
- Because this relation is a partial ordering, it is reflexive, and its directed graph has loops at all vertices.
- Consequently, we do not have to show these loops because they must be present;
- In Figure (b) loops are not shown. Because a partial ordering is transitive, we do not have to show those edges that must be present because of transitivity.



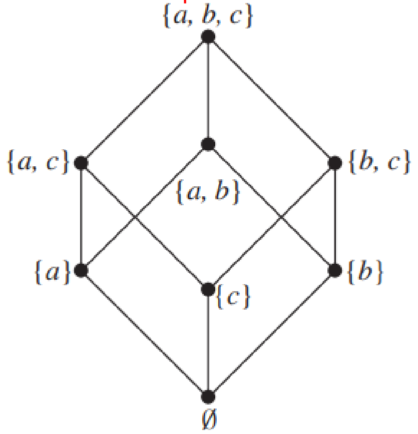
(a) (b) (c)

- In Figure (c) the edges $(1, 3)$, $(1, 4)$, and $(2, 4)$ are not shown because they must be present.
- If we assume that all edges are pointed "upward", we do not have to show the directions of the edges; Figure (c) does not show directions
- The resulting diagram is called the Hasse diagram of $(S, \leq)$, named after the twentieth-century German mathematician Helmut Hasse.
- Let $(S, \leq)$ be a poset. We say that an element $y \in S$ **covers** an element $x \in S$ if $x \prec y$ and there is no element $z \in S$ such that $x \prec z \prec y$.

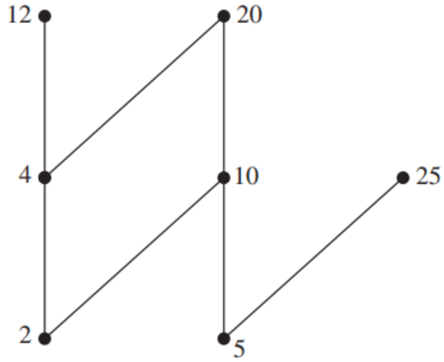
- The set of pairs (x, y) such that y covers x is called the **covering relation** of $(S, \leq)$.
- **EXAMPLE 10** Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.
- *Solution:* Remove all loops, as shown in Figure (b). Then delete all the edges implied by the transitive property. These are $(1, 4)$, $(1, 6)$, $(1, 8)$, $(1, 12)$, $(2, 8)$, $(2, 12)$, and $(3, 12)$.
- Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram.
- The resulting Hasse diagram is shown in Figure (c).



- **EXAMPLE 11** Draw the Hasse diagram for the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set $P(S)$ where $S = \{a, b, c\}$.

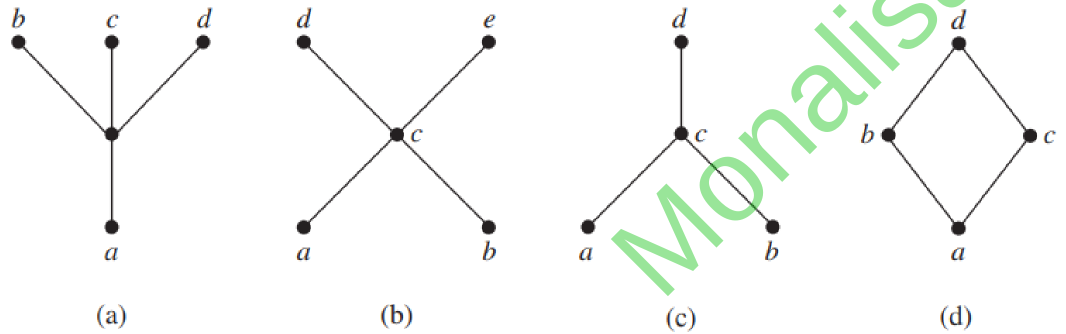


- **Maximal and Minimal Elements**
- An element of a poset is called maximal if it is not less than any element of the poset.
- That is, a is **maximal** in the poset $(S, \leq)$ if there is no $b \in S$ such that $a \prec b$.
- Similarly, an element of a poset is called minimal if it is not greater than any element of the poset.
- That is, a is **minimal** if there is no element $b \in S$ such that $b \prec a$.
- Maximal and minimal elements are easy to spot using a Hasse diagram.
- They are the "top" and "bottom" elements in the diagram.
- **EXAMPLE 12** Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, \mid)$ are maximal, and which are minimal?



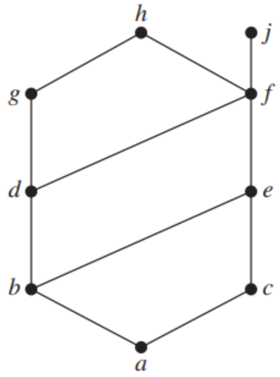
- *Solution:* The Hasse diagram in Figure for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5.
- a poset can have more than one maximal element and more than one minimal element.

- Sometimes there is an element in a poset that is greater than every other element.
- Such an element is called the greatest element.
- That is, a is the **greatest element** of the poset $(S, \leq)$ if $b \leq a$ for all $b \in S$.
- The greatest element is unique when it exists .
- An element is called the least element if it is less than all the other elements in the poset.
- That is, a is the **least element** of $(S, \leq)$ if $a \leq b$ for all $b \in S$.
- The least element is unique when it exists.
- **EXAMPLE 13** Determine whether the posets represented by each of the Hasse diagrams in Figure have a greatest element and a least element.



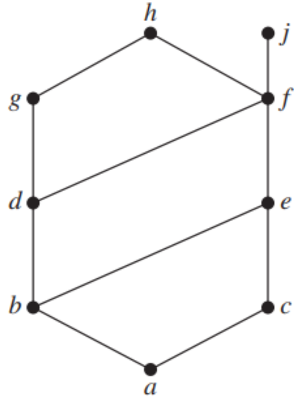
- *Solution:* (a)The least element of the poset is a no greatest element.
- (b) The poset has neither a least nor a greatest element.
- (c) The poset has no least element greatest element is d.
- (d) The poset has least element a and greatest element d.

- **EXAMPLE 14** Let S be a set. Determine whether there is a greatest element and a least element in the poset $(P(S), \subseteq)$.
- *Solution:* The least element is the empty set, because $\emptyset \subseteq T$ for any subset T of S.
- The set S is the greatest element in this poset, because $T \subseteq S$ whenever T is a subset of S.
- Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset A of a poset $(S, \leq)$.
- If u is an element of S such that $a \leq u$, $\forall a \in A$, then u is called an **upper bound** of A.
- Likewise, there may be an element less than or equal to all the elements in A.
- If l is an element of S such that $l \leq a$, $\forall a \in A$, then l is called a **lower bound** of A.
- **EXAMPLE 15** Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure.



- *Solution:* The upper bounds of $\{a, b, c\}$ are e, f, j, and h, and its only lower bound is a.
- There are no upper bounds of $\{j, h\}$, and its lower bounds are a, b, c, d, e, and f.
- The upper bounds of $\{a, c, d, f\}$ are f, h, and j, and its lower bound is a.

- The element x is called the **least upper bound** of the subset A if x is an upper bound that is less than every other upper bound of A.
- That is, x is the least upper bound of A if $a \leq x$ whenever $a \in A$, and $x \leq z$ whenever z is an upper bound of A.
- Similarly, the element y is called the **greatest lower bound** of A if y is a lower bound of A and $z \leq y$ whenever z is a lower bound of A.
- The greatest lower bound and least upper bound of A is unique if it exists.
- The greatest lower bound and least upper bound of a subset A are denoted by glb(A) and lub(A), respectively.
- **EXAMPLE 16** Find the greatest lower bound and the least upper bound of $\{b, d, g\}$, if they exist, in the poset shown in Figure.

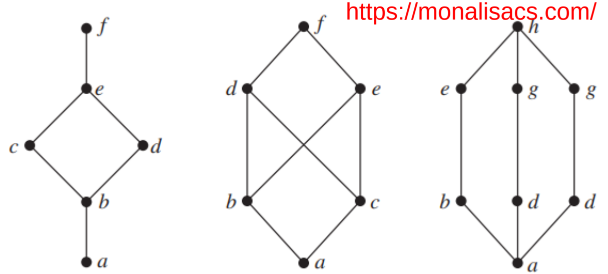


- *Solution:* The upper bounds of $\{b, d, g\}$ are g and h. Because $g \prec h$, g is the least upper bound.
- The lower bounds of $\{b, d, g\}$ are a and b. Because $a \prec b$, b is the greatest lower bound.
- **EXAMPLE 17** Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

- *Solution:* An integer is a lower bound of {3, 9, 12} if 3, 9, and 12 are divisible by this integer.
- The only such integers are 1 and 3.
- Because 1 | 3, 3 is the greatest lower bound of {3, 9, 12}.
- The only lower bound for the set {1, 2, 4, 5, 10} with respect to | is the element 1.
- Hence, 1 is the greatest lower bound for {1, 2, 4, 5, 10}.
- An integer is an upper bound for {3, 9, 12} if and only if it is divisible by 3, 9, and 12.
- Which is 36. Hence, 36 is the least upper bound of {3, 9, 12}.
- A positive integer is an upper bound for the set {1, 2, 4, 5, 10} if and only if it is divisible by 1, 2, 4, 5, and 10.
- Which is 20. Hence, 20 is the least upper bound of {1, 2, 4, 5, 10}.

Lattices

- A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**



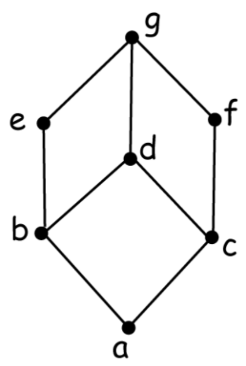
(a) (b) (c)

- **EXAMPLE 18** Determine whether the posets represented by each of the Hasse diagrams in Figure are lattices.
- *Solution:* The posets represented by the Hasse diagrams in (a) and (c) are both lattices because in each poset every pair of elements has both a least upper bound and a greatest lower bound.
- The poset with the Hasse diagram shown in (b) is not a lattice, because the elements *b* and *c* have no least upper bound.
- Each of the elements *d*, *e*, and *f* is an upper bound, but none of these three are lub.
- **EXAMPLE 19** Is the poset $(\mathbf{Z}^+, |)$ a lattice?
- *Solution:* Let *a* and *b* be two positive integers. The least upper bound and greatest lower bound of these two integers are the least common multiple and the greatest common divisor of these integers, respectively. It follows that this poset is a lattice.
- **EXAMPLE 20** Determine whether the posets $(\{1,2,3,4,5\}, |)$ and $(\{1,2,4,8,16\}, |)$ are lattices.
- *Solution:* Because 2 and 3 have no upper bounds in $(\{1, 2, 3, 4, 5\}, |)$, they certainly do not have a least upper bound. Hence, the first poset is not a lattice.
- Every two elements of the second poset have both a least upper bound and a greatest lower bound. Hence, this second poset is a lattice.

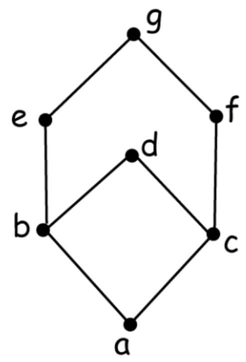
- **EXAMPLE 21** Determine whether *(P (S), ⊆)* is a lattice where *S* is a set.
- *Solution:* Let *A* and *B* be two subsets of *S*. The least upper bound and the greatest lower bound of *A* and *B* are $A \cup B$ and $A \cap B$, respectively. Hence, *(P (S), ⊆)* is a lattice.
- A lattice can be described using two binary operations : *join* and *meet*.
- The join, or sum, is the least upper bound (LUB), sometimes called the supremum or Sup.
- And the meet, or product, of two elements, is the greatest lower bound (GLB), sometimes called the infimum or Inf.

Join (or sum): "a join b"	$LUB(a,b) = a \vee b$
Meet (or product): "a meet b"	$GLB(a,b) = a \wedge b$

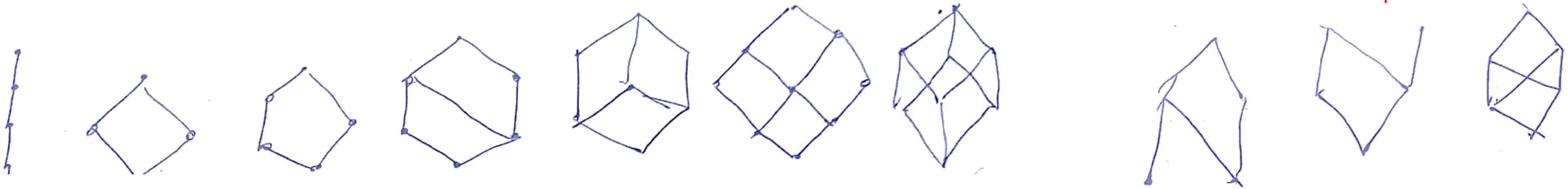
- **EXAMPLE 22** let's determine if the following posets are lattice using a Hasse diagram.
- *Solution:* The left figure is a lattice because each pair of elements has both a lub(join) and glb(meet).
- However, the right figure is not a lattice because each pair of elements are incomparable.
- The upper bound for b and c is {d,e,f,g}, we can't identify which one of these vertices is the join(LUB) — therefore, this poset is not a lattice.



Lattice



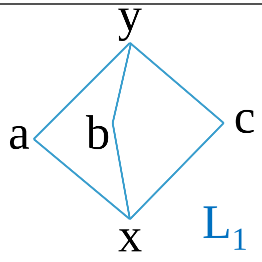
Not a Lattice

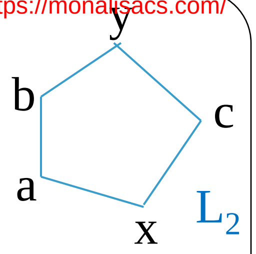


- First 7 Hesse diagram are lattice while last 3 are not lattice.
- Several types of lattices :
- Complete Lattice – all subsets of a poset have a join and meet, such as the divisibility relation for the natural numbers or the power set with the subset relation.
- Bounded Lattice – if the lattice has a least and greatest element, denoted 0 and 1 respectively.
- Complemented Lattice – a bounded lattice in which every element is complemented. Namely, the complement of 1 is 0, and the complement of 0 is 1.
- Distributive Lattice – if for all elements in the poset the distributive property holds.
- Boolean Lattice – a complemented distributive lattice, such as the power set with the subset relation.
- The total order set is always Distributive Lattice .

❖ A lattice L said to be a distributive lattice if every element in L has “atmost one complement”

- **EXAMPLE 23** Which of the following is/are distributive lattice ?





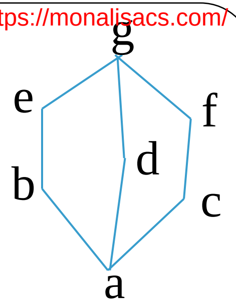
- L_1: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \Rightarrow a \vee x = y \wedge y$
- $a \neq y$ not a distributive lattice
- L_2: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \Rightarrow a \vee x = b \wedge y$
- $a \neq b$ not a distributive lattice
- **EXAMPLE 24** Which of the following is not a distributive lattice

A. (p(A); ⊆) where A={a,b,c,d}
- Distributive law hold for any 3 subset hence its distributive.

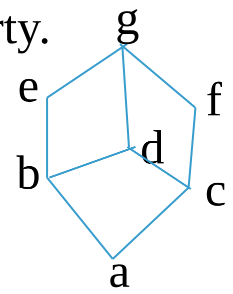
B. $(D_{64};/)$ $D_{64} = \{1,2,4,8,16,32,64\}$
- It's a total order set hence distributive lattice.
- **EXAMPLE 25** In the lattice $(D_{18};/)$ which of the following is true

A. The complement of 1 is 18. true
B. The complement of 2 is 9. true
C. The complement of 3 is 6. false
D. The complement of 6 doesn't exist. true
- It is a distributive lattice but not complemented lattice.

- **EXAMPLE 26** For the lattice shown below how many complement 'e' had ?



- A. 1 B.2 C.3 D.4
- It's a complementary lattice but not a distributive lattice
- Complement of d=b,e,c,f Complement of b=d,c,f
- Complement of c=d,b,e Complement of e=d,c,f
- Ans: 3
- **EXAMPLE 27** For the lattice shown check distributive and complemented property.



- The lattice is neither distributive nor complemented.
- Complement of a=g
- Complement of b=f
- Complement of c=e
- Complement of d=does not exist
- Complement of e=f,c
- Its not complemented ,also not distributive as for e we have 2 complement .

2.12 Groups Theory

- **Algebraic Structure (S,*):** A nonempty set S is called a algebraic structure with respect to binary operation '*'.
- If (a*b) ∈ S , ∀a,b ∈ S then * is a closer operation on S.
- N={1,2,3,…..∞}
- Z={Set of all integer }
- Q={Set of all rational numbers}
- R={Set of all real numbers}
- C={Set of all complex numbers}
- **EXAMPLE 1**
- (N ,+, ×) is an algebraic structure w.r.t '+, ×'.
- (N , −) is not an algebraic structure w.r.t ' − '.
- (N ,/) is not an algebraic structure w.r.t '/'.
- (Z ,+,×,-) is an algebraic structure w.r.t '+,×,-'.
- (Z ,/) is not an algebraic structure w.r.t '/'.

- **Semigroup:** An algebraic structure (S,*) is a semigroup if ((a*b)*c)=(a*(b*c)) , ∀a,b,c ∈ S.
- i.e * is associative on S.
- **EXAMPLE 2**
- (N ,+) is a semigroup as '+' is associative .
- (N , ×) is a semigroup as '×' is associative .
- (N , −) is not a semigroup as '−' is neither associative nor a closer operation.
- (N , /) is not a semigroup as '/' is neither associative nor a closer operation.
- **Monoid:** A semigroup is called a monoid if identity element exist.
- Let e ∈ S s.t. (a*e)=a for ∀a ∈ S then e is identity element w.r.t *.
- **EXAMPLE 3**
- (N,×) is a monoid as a × 1=a for ∀a ∈ N.
- (N,+) is not a monoid as a+0=a and 0 ∉ N.
- (Z,+) is a monoid as 0 ∈ Z.
- (Z,-) is not a monoid as '–' is not associative ,hence not semigroup not a monoid.

- **Group:** A monoid (S,*) is called a group if for each a ∈ S there exist an element b ∈ S s.t. (a*b)=(b*a)=e. b= inverse of a=a^{-1} .
- **EXAMPLE 4** (Z ,+) is a group a+(-a)=0
- (N , ×) is not a group a× $\frac{1}{a}$=1 , $\frac{1}{a}$ ∉ N
- **Abelian group:** A group (G,*) is said to be abelian if (a*b)=(b*a) , ∀a,b ∈ S.
- A group with commutative property.
- **EXAMPLE 5** (Z ,+) is an abelian group

Closer	=Algebraic structure
Closer , Associative	=Semigroup
Closer , Associative , identity	=Monoid
Closer , Associative , identity ,Inverse	=Group
Closer , Associative , identity ,Inverse, Commutative	=Abelian group

- **EXAMPLE 6** Set of all non singular matrices of order (2×2) is a group w.r.t matrix multiplication but not an abelian group because matrix multiplication is not commutative .

- **EXAMPLE 7** Which of the following is not a semigroup ?
- A) {1,3,5,……∞} w.r.t ‘×’ .It’s closer and associative hence semigroup .
- B) {2,4,6,8…. ∞} w.r.t ‘+’.semigroup
- C) {1,3,5,7,….∞} w.r.t ‘+’. It’s not closer as odd +odd =even hence not semigroup.
- D) {2,4,6,8….∞} w.r.t ‘×’. semigroup .
- **EXAMPLE 8** The set A={0≤x ≤1 and x is a real number}w.r.t ‘×’ is _________ .
- A)A Semigroup but not a Monoid B)A Monoid but not a Group
- C)A group D) Not a Semigroup
- Multiplication is closer ,Associative ,have identity 1 w.r.t ‘×’ .
- But no inverse as $\frac{2}{3} \times \frac{3}{2} = 1$, $\frac{3}{2} > 1$
- Hence a Monoid but not a Group
- **EXAMPLE 9** Let Z=Set of all int ,(a*b)=Min of {a,b}.Then (Z,*) is _______ .
- It is closer ,Associative but no identity .Min(a,e)=a ,e>a but e is not unique.
- Hence it’s a Semigroup .

- **Finite groups:** If number of elements in a group is finite then it is called as finite group .
- Order of finite group =number of elements in the group .
- If a group has only one element then that element is identity element of group .
- S={0} is a group w.r.t '+'
- S={1} is a group w.r.t '×'
- S={1,-1} is a group w.r.t '×'
- **EXAMPLE 10** The cube roots of unity $\{1,w,w^2\}$ is a group w.r.t. '×'
- Inverse of $1=1, w= w^2, w^2 = w$
- **EXAMPLE 11** The forth roots of unity {1,-1,i,-i} is a group w.r.t. '×'
- $i^2=-1$,Inverse of 1=1,-1=-1,-i=i,i=-i
- Addition module $\oplus_m$,Multiplication module $\otimes_m$
- Where m is a positive int ,If a,b are any two positive integers
- $a \oplus_m b = a+b$ if $a+b < m$, $=\text{reminder } \frac{a+b}{m}$ if $a+b \geq m$
- $a \otimes_m b = a \times b$ if $a \times b < m$, $=\text{reminder } \frac{a \times b}{m}$ if $a+b \geq m$

×	1	w	w^2
1	1	w	w^2
w	w	w^2	1
w^2	w^2	1	w

×	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

- **EXAMPLE 12** S={2,4,6,8} is a group $\otimes_{10}$
- Closer , Associative , identity ,Inverse hence it's a group
- Identity element =6
- Inverse of 2=8, 4=4,6=6,8=2

	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

- **EXAMPLE 13** Which of the following is a group
- A){1,2,3,4,5} w.r.t $\oplus_6$ B) {1,2,3,4,5} w.r.t$\otimes_6$
- C) {0,1,2,3,4,5} w.r.t$\otimes_6$ D) {1,2,3,4,5,6} w.r.t$\otimes_7$
- A,B are not closer hence not group.C have no inverse hence not group .D is a group
- **Order of an element O(a)** Let (G,*) be a group with identity element e ,for any element a∈G order of a O(a)=n ,where n is the smallest +ve integer s.t. a^n=e.
- **EXAMPLE 14** G={1,-1} is a group w.r.t multiplication O(1)=1,O(-1)=2
- **EXAMPLE 15** G={1,w,w^2} is a group w.r.t multiplication
- O(1)=1,O(w)=3, O(w^2)=3
- **EXAMPLE 16** G={1,-1,i,-i} is a group w.r.t multiplication
- O(1)=1,O(-1)=2, O(i)=4, O(-i)=4

- **Subgroup:** Let (G,*) be a group. A subset H ,H⊆G is called a subgroup of G if H is a group w.r.t. *.
- Let (G,*) be a group with identity element e ,G & {e} are called *trivial subgroup* of G.
- **Proper Subgroup:** Any other subgroup of G which is not a trivial called proper subgroup .
- If G is Abelian, then a subgroup of G should be abelian.
- **EXAMPLE 17** G={1,-1,i,-i} is a group w.r.t multiplication ,H={1,-1} is a proper subgroup
- **Lagrange theorem** **F**or any finite group say G, the order of subgroup H of group G divides the order of G. The order of the group represents the number of elements.
- This theorem was given by Joseph-Louis Lagrange.
- **EXAMPLE 18** G={0,1,2,3,4,5} is a group w.r.t $\oplus_6$ .Which of the fallowing subset of G are subgroup of G .
- A)H_1={0,3} B) H_2={0,5} C) H_3={0,2,4} D) H_4={0,2,3,4}
- A) Subgroup B) not closer hence not subgroup
- C) Subgroup D) 4 is not divisible by 6. Hence not subgroup.
- **EXAMPLE 19** G={1,2,3,4,5,6} is a group w.r.t $\otimes_7$ Which of the fallowing are subgroup of G.
- A) H_1={1,3} B) H_2={1,6} C) H_3={1,2,4} D) H_4={1,3,5}
- A) Not closer hence not subgroup B) Subgroup
- C) Subgroup D) Not closer hence not subgroup

- **EXAMPLE 20** G={1,3,5,7} is a group w.r.t $\otimes_8$ Which of the fallowing are subgroup of G.
- A) H_1={1,3} B) H_2={1,5} C) H_3={1,7} D) H_4={1,3,5}
- A,B,C are subgroup while D is not a subgroup as 3 is not divisible by 4.
- Intersection of any two subgroup of a group is also a subgroup .
- Union of two subgroup H_1 & H_2 of a group G is also a subgroup of G iff $H_1 \subset H_2$ or $H_2 \subset H_1$.
- **EXAMPLE 21** Let (G,*) be a group of order p where p is a prime number ,how many proper subgroup are possible for G ? A)0 B)2 C)p-2 D)p
- Order(G)=p,Order(H)=1 or p
- H={e} or{G}
- Hence G has no proper subgroup . Ans (A)0
- **Cyclic Group** A group (G,*) is said to be cyclic if there exist an element a∈G s.t. every element of G can be written as a^n for some integer n.
- The element a is called generating element or generator .
- Most of finite group are cyclic group .
- **EXAMPLE 22** The set {1,-1} is a cyclic group w.r.t multiplication.
- Generator =-1,$(-1)^2=1$
- **EXAMPLE 23** The set {1,w,w^2} is a cyclic group w.r.t multiplication.
- Generator =w,w^2,$(w)^3=1$, $(w^2)^2=w$, $(w^2)^3=1$

- If (G,*) is a cyclic group with generator ‘a’ then
- (1) a^{-1} is also a generator of G
- (2) O(a)=O(G)
- **EXAMPLE 24** G={0,1,2,3} is a cyclic group w.r.t $\oplus_4$
- $1^2= 1 \oplus_4 1=2$, $1^3= 1 \oplus_4 1 \oplus_4 1=3$, $1^4= 1\oplus_4 1\oplus_4 1\oplus_4 1=0$
- $3^2= 3 \oplus_4 3=2$, $3^3= 3 \oplus_4 3\oplus_4 3=1$, $3^4= 3 \oplus_4 3\oplus_4 3\oplus_4 3=0$
- Generator=1,3
- **EXAMPLE 25** G={1,2,3,4} is a cyclic group w.r.t $\otimes_5$ .Generator=2,3
- **EXAMPLE 26** G={1,3,5,7} is the only group w.r.t $\otimes_8$ not a cyclic group so we can’t find generator
- **Theorem** – If a Group order is prime number, then it is cyclic group and every Cyclic group is abelian group.
- Number of generator in G=Ø(n) [Euler function of n]
- Counts the number of positive integers less than n that are coprime to n.
- **If the prime factorization of n is given by n =$p_1^{e_1}$*...*$p_n^{e_n}$,**
- **Then $\varphi(n) = n *(1 - 1/p_1)* ... (1 - 1/p_n)$.**
- **EXAMPLE 27** (G ,*) is a cyclic group of O(8) ,Number of generator in G=4
- $8=2^3$, Ø(8)=8*(1-1/2)=4
- Coprime to 8={1,3,5,7}

- **EXAMPLE 28** if (G,*) is a cyclic group of order 100 then number of generator of G= Ø(100)
- $100=2^2*5^2$, Ø(100)=100*(1-1/2)(1-1/5)=100*(1/2)*(4/5)=40
- **EXAMPLE 29** (G,*) is given below

*	a	b	c	d
a	b	d	a	c
b	d	c	b	a
c	a	b	c	d
d	c	a	d	b

- The generators are (a)a&b (b)a&c (c)a&d (d)b&d
- Identity element=c ,so c can't be generator
- a^2=b, a^3=d, a^4=c , a is a generator .
- b^2=c, b^3=b, b is not a generator .
- d^2=b, d^3=a, d^4=c , d is a generator
- Ans : a,d
- **EXAMPLE 30** The incomplete composition table of finite group is given below.
- The last row of the table is

*	a	b	c	d
a	b	d	a	c
b	d	c	b	a
c	a	b	c	d
d				

- (a)a b c d
- (b)c a d b
- (c)a b d c
- (d)c a b d

- For cyclic group fallowing statements are true
- Every cyclic group is abelian
- Every group of prime order is cyclic & abelian
- Every subgroup of cyclic is cyclic but generator of subgroup may not be same as cyclic group.
- For group fallowing statements are true
- In a group (G,*) with identity element e if a*a=a then a=e
- $a*a=a*e \Rightarrow a=e$
- In a group(G,*) with identity element e if $a^{-1}=a$, $\forall a \in G$,then G is abelian group.
- In a group (G,*) if $(a*b)^2=a^2*b^2$, $\forall a,b \in G$ then G is a abelian group