Discrete Mathematics

Chapter 4 : Combinatorics

GATE CS Lectures
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Section1: Engineering Mathematics

- **Discrete Mathematics**: Propositional and first order logic. Sets, relations, functions, partial orders and lattices. Monoids, Groups. Graphs: connectivity, matching, coloring. Combinatorics: counting, recurrence relations , generating functions.
- **Linear Algebra**: Matrices, determinants, system of linear equations, eigenvalues and eigenvectors, LU decomposition.
- **Calculus**: Limits, continuity and differentiability. Maxima and minima. Mean value theorem. Integration.
- **Probability and Statistics**: Random variables. Uniform, normal, exponential, poisson and binomial distributions. Mean, median, mode and standard deviation. Conditional probability and Bayes theorem.

- **Discrete Mathematics**: Propositional and first order logic. Sets, relations, functions, partial orders and lattices.Monoids, Groups.Graphs: connectivity, matching, coloring.Combinatorics : counting, recurrence relations , generating functions.
- Chapter 1: Logic
- Propositional Logic, Propositional Equivalences , Predicates and Quantifiers , Nested Quantifiers , Rules of Inference , Introduction to Proofs.
- Chapter 2 : Set Theory
- Sets, relations, functions, partial orders and lattices.Monoids, Groups.
- Chapter 3 : Graph Theory
- Graphs: connectivity, matching, coloring.
- Chapter 4 : Combinatorics
- Counting, Recurrence relations , Generating functions

- **Chapter 4 : Combinatorics**
- 4.1 The Basics of Counting
- 4.2 Permutations and Combinations
- 4.3 Binomial Coefficients and Identities
- 4.4 The Pigeonhole Principle
- 4.5 Inclusion–Exclusion
- 4.6 Recurrence Relations
- 4.7 Generating Functions

Basic Counting Principles

- **THE PRODUCT RULE** Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.
- The product rule applies when a procedure is made up of separate tasks
- **EXAMPLE 1** The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?
- *Solution:* The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers.
- $26 \cdot 100 = 2600$ different ways that a chair can be labeled.
- An extended version of the product rule: Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \ldots, T_m$ in sequence. If each task T_i, $i = 1, 2, \ldots, n$, can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 \cdots n_m$ ways to carry out the procedure.

- **EXAMPLE 2** How many different bit strings of length seven are there?
- *Solution:* Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1.
- $2^7 = 128$ different bit strings of length seven.
- **EXAMPLE 3** How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits.
- *Solution:* There are 26 choices for each of the three uppercase English letters and ten choices for each of the three digits.
- $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ possible license plates.
- **EXAMPLE 4 Counting Functions** How many functions are there from a set with m elements to a set with n elements?
- *Solution:* A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain.
- $n \cdot n \cdot \cdots \cdot n = n^m$ functions from a set with m elements to one with n elements.
- For example, $5^3 = 125$ different functions from a set with three elements to a set with five elements.
- **EXAMPLE 5 Counting One-to-One Functions** How many one-to-one functions are there from a set with m elements to one with n elements?

- *Solution:* First note that when $m > n$ there are no one-to-one functions from a set with m elements to a set with n elements.
- Now let $m \leq n$.
- Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$.
- There are n ways to choose the value of the function at a_1.
- Because the function is one-to-one, the value of the function at a_2 can be picked in $n - 1$ ways.
- In general, the value of the function at a_k can be chosen in $n - k + 1$ ways.
- There are $n(n - 1)(n - 2) \cdots (n - m + 1)$ one-to-one functions from a set with m elements to one with n elements.
- For example, there are $5 \cdot 4 \cdot 3 = 60$ one-to-one functions from a set with three elements to a set with five elements.
- THE SUM RULE If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.
- **EXAMPLE 6** A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

- *Solution:* The student can choose a project by selecting a project from the first list, the second list, or the third list.
- By the sum rule there are 23 + 15 + 19 = 57 ways to choose a project.
- **EXAMPLE 7 A** pair of dice were tossed number of ways we get total of 7 or 8 is
- *Solution:* 7= {(1,6) ,(2,5) ,(3,4) ,(4,3) ,(5,2) ,(6,1)}= 6 ways
- 8={(2,6) ,(3,5) ,(4,4) ,(5,3) ,(6,2)}=5 ways
- Total number of ways=6+5=11
- If the dice are not distinguished
- 7= {(1,6) ,(2,5) ,(3,4)}= 3 ways
- 8={(2,6) ,(3,5) ,(4,4) }= 3 ways
- Total number of ways=3+3=6
- **EXAMPLE 8 H**ow many int between 10^5 and 10^6 have no digits other than 0,2,5,8
- *Solution:* _(2,5,8) _(0,2,5,8) _(0,2,5,8) _(0,2,5,8) _(0,2,5,8) _(0,2,5,8)
- 3*4*4*4*4*4=3072

- **EXAMPLE 9** What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?
- $k := 0$
- **for** $i_1 := 1$ **to** n_1
-     $k := k + 1$
- **for** $i_2 := 1$ **to** n_2
-     $k := k + 1$
- ...
- **for** $i_m := 1$ **to** n_m
-     $k := k + 1$
- *Solution:* Each time a loop is traversed, 1 is added to k.
- We only traverse one loop at a time, the sum rule shows that the final value of k, is the number of ways to traverse one of the m loops is
- $n_1 + n_2 + \cdots + n_m$.

- **EXAMPLE 10** What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?
- $k := 0$
- **for** $i_1 := 1$ **to** n_1
-     **for** $i_2 := 1$ **to** n_2
-         . . .
-         **for** $i_m := 1$ **to** n_m
-             $k := k + 1$
- *Solution:* Each time the nested loop is traversed, 1 is added to k.
- By the product rule, it follows that the nested loop is traversed $n_1 n_2 \cdots n_m$ times.
- Hence, the final value of k is $n_1 n_2 \cdots n_m$.

- **EXAMPLE 11** 4 distinct dice were tossed ,Number of outcome in which at least one dice show 2=?
- *Solution:* Number of outcomes possible with four dice= 6^4
- Number of outcomes possible in which no dice shows 2 = 5^4
- Required number of outcome = $6^4 – 5^4$ =671
- **EXAMPLE 12** How many four digits integers are there with digits 6 appearing exactly once ?
- *Solution:* Case 1 : _(6) _(0-5,7-9) _(0-5,7-9) _(0-5,7-9)
- Number of 4 digits integers with 6 at first place =9*9*9
- Case 2: _(1-5,7-9) _(6) _(0-5,7-9) _(0-5,7-9)
- Or _(1-5,7-9) _(0-5,7-9)_(6) _(0-5,7-9)
- Or _(1-5,7-9) _(0-5,7-9) _(0-5,7-9) _(6)
- Number of 4 digits integers with 6 at 2nd ,3rd ,4th place =8*9*9+8*9*9+8*9*9
- Total number of four digits integers with 6 appearing exactly once
- =9*9*9 + 8*9*9 + 8*9*9 + 8*9*9
- =2673

4.2 Permutations and Combinations

- A **permutation** of a set of distinct objects is an ordered arrangement of these objects.
- An ordered arrangement of r elements of a set is called an ***r*-permutation**.
- **THEOREM 1** If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are
- $P(n, r) = n(n - 1)(n - 2) \cdot \cdot \cdot (n - r + 1)$ r-permutations of a set with n distinct elements.
- **COROLLARY 1** If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \frac{n!}{(n-r)!}$
- **EXAMPLE 1** In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?
- *Solution:* There are five ways to select the first student to stand at the start of the line.
- There are four ways to select the second student in the line.
- After the first and second students have been selected, there are three ways to select the third student in the line.
- By the product rule, there are $5 \cdot 4 \cdot 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.
- There are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to arrange all five students in a line.

- **EXAMPLE 2** How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?
- *Solution:* $P(100, 3) = 100 \cdot 99 \cdot 98 = 970{,}200$
- **EXAMPLE 3** Suppose that there are eight runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?
- *Solution:* $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$ possible ways to award the medals.
- **EXAMPLE 4** Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?
- *Solution:* The number of possible paths between the cities is the number of permutations of seven elements, because the first city is determined, but the remaining seven can be ordered arbitrarily.
- $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ ways for the saleswoman to choose her tour.

- **EXAMPLE 5** How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?
- *Solution:* Because the letters *ABC* must occur as a block,
- The number of permutations of six objects, namely, the block *ABC* and *D*, *E*, *F* , *G*, and *H*.
- There are 6! = 720 permutations of the letters *ABCDEFGH* in which *ABC* occurs as a block.
- **EXAMPLE 6** How many ways 15 different books can be distributed among 10 persons so that no person can take more than one book and maximum number of books have to be distributed ?
- *Solution:* P(15,10)=10897286400
- **EXAMPLE 7** How many ways 6 person can sit in a row ?
- *Solution:* 6!=720
- **EXAMPLE 8** How many ways 6 persons (A,B,C,D,E,F) can sit in a row so that A & B are not side by side ?
- *Solution:* Number of way 6 person can sit 6!=720 ways
- If a and b are sitting side by side then a &b consider as 1 .
- (a,b) ,c,d,e,f can sit =5!=120 ways
- a & b can sit in 2 ways =(ab),(ba)
- Hence number of ways A & B are not side by side = 720-(120*2)=720-240=480 ways

- **EXAMPLE 9** How many ways 5 boys and 5 girls can sit in a row so that no two boys are sitting side by side ?
- *Solution:* Girls can sit in a row 5! Way .
- Now there are 6 places for boys , So boys can sit P(6,5) ways .
- Total number of ways=5!*P(6,5)=120*720=86400 ways.
- **Circular Permutations** Number of permutation of n distinct objects around a circle =(n-1)!
- **EXAMPLE 10** How many ways 5 boys and 5 girls can sit around a circle table so that no two boys can sit side by side ?
- *Solution:* Girls can sit in 4! Way
- Now We have 5 different place among girls for boys to sit ,5!
- Total number of ways= 4!*5!=24*120=2880
- **Permutations with Repetitions** Number of permutations of n distinct objects taken k at a time with unlimited repetitions $=n^k$
- **EXAMPLE 11** How many ways 5 letter permutation are possible with letter {a,b,c,d} *Solution:* 4^5=1024

- Suppose we have n objects, of which n_1 objects are alike ,n_2 objects are alike ……n_k objects are alike.
- Then number of permutation of n objects taken one at a time = $\frac{n!}{n_1!n_2!\ldots n_k!}$
- **EXAMPLE 12** How many 10 letter permutations are possible with the letter of the set {a , a , b , b , b , c , c , c , c} if all the letters of this set are used at a time ?
- *Solution:* $\frac{10!}{2!*3!*4!}$ =12600
- **EXAMPLE 13** How many binary sequence of length 10 are possible with 6 ones and 4 zeros?
- *Solution:* $\frac{10!}{6!*4!}$ = 210
- **EXAMPLE 14** A coin is tossed 10 times number of outcome with 5 heads and 5 tails are ?
- *Solution:* $\frac{10!}{5!*5!}$ = 252

- A **Combinations** is an unordered selections of objects .
- An ***r*-combination** of elements of a set is an unordered selection of *r* elements from the set.
- **EXAMPLE 15** Let *S* be the set {1, 2, 3, 4}. Then {1, 3, 4} is a 3-combination from *S*.
- (Note that {4, 1, 3} is the same 3-combination as {1, 3, 4}, because the order in which the elements of a set are listed does not matter.)
- The number of *r*-combinations of a set with *n* distinct elements is denoted by *C(n, r)*.
- *C(n, r)* is also denoted by $\binom{n}{r}$ and is called a **binomial coefficient**.
- **EXAMPLE 16** We see that *C*(4, 2) = 6, because the 2-combinations of {*a*, *b*, *c*, *d*} are the six subsets {*a*, *b*}, {*a*, *c*}, {*a*, *d*}, {*b*, *c*}, {*b*, *d*}, and {*c*, *d*}.
- **THEOREM 2** The number of *r*-combinations of a set with *n* elements, where *n* is a nonnegative integer and *r* is an integer with $0 \le r \le n$, equals $C(n,r)=\frac{n!}{r!(n-r)!}$
- ***Proof:*** The *P (n, r) r*-permutations of the set can be obtained by forming the *C(n, r) r* -combinations of the set, and then ordering the elements in each *r*-combination, which can be done in *P (r, r)* ways. $P(n, r) = C(n, r) \cdot P(r, r)$
- This Implies that $C(n,r)=\frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$

- **EXAMPLE 17** How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?
- *Solution:* Because the order in which the five cards are dealt from a deck of 52 cards does not matter, there are
- $C(52, 5) = \frac{52!}{5!(52-5)!} = \frac{52!}{5!*47!}$
- $C(52, 5) = \frac{52*51*50*49*48}{5*4*3*2*1}$
- $C(52, 5) = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2{,}598{,}960.$
- Consequently, there are 2,598,960 different poker hands of five cards that can be dealt from a standard deck of 52 cards.
- $C(52, 47) = \frac{52!}{47!(52-47)!} = \frac{52!}{47!*5!}$
- $C(52, 47) = C(52, 5)$
- **COROLLARY 2** Let n and r be nonnegative integers with $r \le n$. Then $C(n, r) = C(n, n - r)$
- **EXAMPLE 18** How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?
- *Solution:* The number of 5-combinations of a set with 10 elements.
- $C(10, 5) = \frac{10!}{5!(10-5)!} = 252$

- **DEFINITION 1** A *combinatorial proof* of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity. These two types of proofs are called *double counting proofs* and *bijective proofs*, respectively.
- **EXAMPLE 19** A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?
- *Solution:* 6-combinations of a set 30 elements, $C(30, 6)=\frac{30!}{6!(30-6)!}=\frac{30*29*28*27*26*25}{6*5*4*3*2*1}= 593{,}775$
- **EXAMPLE 20** Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?
- *Solution:* $C(9,3)* C(11,4) =\frac{9!*11!}{3!*6!*4!*7!}=\frac{9*8*7*11*10*9*8}{3*2*1*4*3*2*1}= 27{,}720$

- **EXAMPLE 21** How many ways we can select a committee of 5 person out of 5 men and 5 women so that at least 2 women included in the committee ?
- *Solution:* $C(5,2)* C(5,3)+C(5,3)* C(5,2)+C(5,4)* C(5,1)+C(5,5)* C(5,0)$
- $=100+100+25+1=226$
- **EXAMPLE 22** How many binary sequence of length 9 are possible with exactly 3 zeros?
- *Solution:* $C(9,3) = \frac{9!}{3!*6!} = \frac{9*8*7}{3*2*1} = 84$
- **EXAMPLE 23** How many binary sequence of length 10 are possible with exactly 4 zeros and no two zeros are consecutive?
- *Solution:* _1_1_1_1_1_1_ , $C(7,4) = \frac{7!}{4!*3!} = \frac{7*6*5}{3*2*1} = 35$
- **EXAMPLE 24** Suppose n couple are in a party if each person shake hand with every other person except his/her spouse then how many different hand shake are possible ?
- *Solution:* Number of handshake possible with 2n persons =C(2n,2)
- From these n handshake are restricted ,
- So total handshake = $C(2n,2)-n = \frac{2n(2n-1)}{2} - n = 2n(n-1)$

- **Combination with repetition V(n,k)** Number of combination of n distinct objects taken k at a time with unlimited repetition=C(n+k-1,k)
- **EXAMPLE 25** How many ways we can distribute 16 similar balls in 4 boxes so that each box contain at least one ball.
- *Solution:* 4 balls can be place in 1 way .
- Remaining 12 balls have no restriction ,it can distribute V(4,12) way
- $V(4,12)=C(15,12) = \frac{15!}{12!*3!} = \frac{15*14*13}{3*2*1} = 455$
- **EXAMPLE 26** How many +ve solution are possible to the equation $x_1+x_2+x_3+x_4+x_5=20$ so that $(x_1 \geq 2)$, $(x_2 \geq 3)$, $(x_3 \geq 4)$, $(x_4 \geq 6)$, $(x_5 \geq 0)$
- *Solution:* 2+3+4+6+0=15
- $20-15=5$, $V(5,5)=C(9,5) = \frac{9!}{5!*4!} = \frac{9*8*7*6}{4*3*2*1} = 126$
- **EXAMPLE 27** Number of +ve solution to the inequality $x_1+x_2+x_3+x_4+x_5 \leq 10$
- *Solution:* It's same as $x_1+x_2+x_3+x_4+x_5+x_6=10$
- $V(6,10) = C(15,10) = \frac{15!}{10!*5!} = \frac{15*14*13*12*11}{5*4*3*2*1} = 3003$

- **EXAMPLE 28** Number of +ve solution for $x_1+x_2+x_3+x_4+x_5 < 10$
- *Solution:* It's same as $x_1+x_2+x_3+x_4+x_5 \leq 9$
- $x_1+x_2+x_3+x_4+x_5+x_6 = 9$
- $V(6,9) = C(14,9) = \frac{14!}{9!*5!} = \frac{14*13*12*11*10}{5*4*3*2*1} = 2002$
- **EXAMPLE 29** Number of +ve solution for $x_1+x_2+x_3+x_4+x_5 \geq 10$
- *Solution:* Infinite solutions
- **EXAMPLE 30** How many ternary sequence are possible with 6 one's ,6 two's and 4 zero's so that each one is immediately fallowed by two .
- *Solution:* _12_12_12_12_12_12_
- We have 7 places for 4 zeros
- $V(7,4) = C(10,4) = \frac{10!}{4!*6!} = \frac{10*9*8*7}{4*3*2*1} = 210$

4.3 Binomial Coefficients and Identities

- **EXAMPLE 1** $(x + y)^3 = (x + y)(x + y)(x + y) = (xx + xy + yx + yy)(x + y)$
- $= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$
- $= x^3 + 3x^2y + 3xy^2 + y^3$.
- **THEOREM 1** Let *x* and *y* be variables, and let *n* be a nonnegative integer. Then

$$(x+y)^n = \sum_{J=0}^{n} \binom{n}{J} x^{n-J} y^J = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots .. \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

- **EXAMPLE 2** What is the expansion of $(x + y)^4$?
- *Solution:* $(x + y)^4 = \sum_{J=0}^{4} \binom{4}{J} x^{4-J} y^J = \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$
- $= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.
- **EXAMPLE 3** What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?
- *Solution:* coefficient is $\binom{25}{13} = \frac{25!}{13!12!} = 5,200,300$.
- **EXAMPLE 4** What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?
- *Solution:* $(2x + (-3y))^{25} = \sum_{J=0}^{25} \binom{25}{J} (2x)^{25-J} (-3y)^J$.
- When $j = 13$, $\binom{25}{13} (2x)^{12} (-3y)^{13} = \binom{25}{13} 2^{12} (-3)^{13} x^{12} y^{13}$

- **COROLLARY 1** Let *n* be a nonnegative integer. Then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
- ***Proof:*** Using the binomial theorem with *x* = 1 and *y* = 1, we see that
- $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$
- **COROLLARY 2** Let *n* be a positive integer. Then $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$
- ***Proof:*** Using the binomial theorem with *x* = 1 and *y* = -1, we see that
- $0^n = (1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$
- **COROLLARY 3** Let *n* be a nonnegative integer. Then $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$
- ***Proof:*** Using the binomial theorem with *x* = 1 and *y* = 2, we see that
- $3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k$

Pascal's Identity and Triangle

- **THEOREM 2 PASCAL'S IDENTITY** Let n and k be positive integers with $n \geq k$. Then
- $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$
- **Pascal's triangle**: Pascal's identity shows that when two adjacent binomial coefficients in this triangle are added, the binomial coefficient in the next row between these two coefficients is produced.

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$$
$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$$
$$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$$
$$\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}$$

...

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

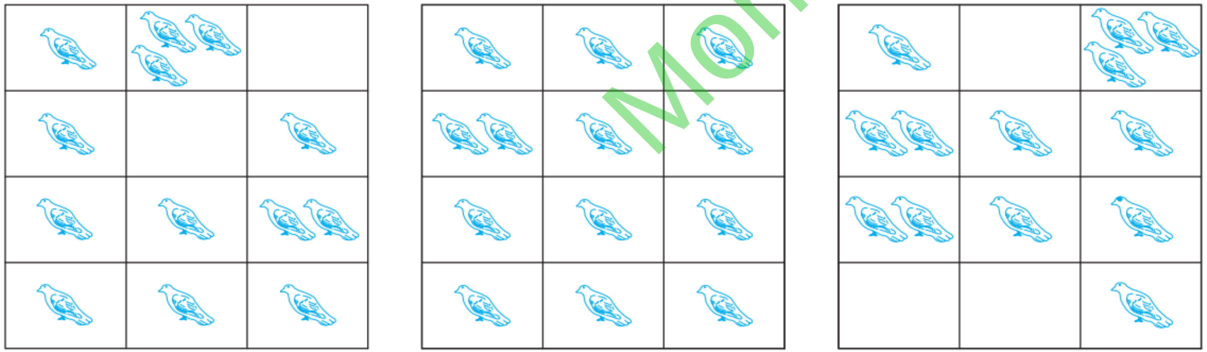
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

...

(b)

4.4 The Pigeonhole Principle

- **THEOREM 1 THE PIGEONHOLE PRINCIPLE** If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.
- The pigeonhole principle is also called the **Dirichlet drawer principle**, after the nineteenth century German mathematician G. Lejeune Dirichlet, who often used this principle in his work.
- **pigeonhole principle**, which states that if there are more pigeons than pigeonholes,then there must be at least one pigeonhole with at least two pigeons in it .
- Of course, this principle applies to other objects besides pigeons and pigeonholes.
-



(a) (b) (c)

- **COROLLARY 1** A function f from a set with $k + 1$ or more elements to a set with k elements is not one-to-one.
- **EXAMPLE 1** Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.
- **EXAMPLE 2** In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.
- **EXAMPLE 3** How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?
- *Solution:* There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.
- **THEOREM 2 THE GENERALIZED PIGEONHOLE PRINCIPLE** If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.
- **EXAMPLE 4** Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

- For any +ve int k if kn+1 pigeons kept in n pigeonhole then A= $\frac{kn+1}{n}$ =k+1/n
- Some pigeonholes contain at least $\lceil A \rceil$=k+1 pigeons .
- Some pigeonholes contain at most $\lfloor A \rfloor$ pigeons
- Suppose we have n pigeonholes the minimum number of pigeon required to ensure that some pigeonhole consists at least k+1 pigeons=kn+1
- **EXAMPLE 5** What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?
- *Solution:* $\lceil N/5 \rceil = 6$=k+1 ,k=5
- The smallest such integer is $N = 5 \cdot 5 + 1 = 26$.
- Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.
- A standard deck of 52 cards has 13 kinds of cards, with four cards of each of kind, one in each of the four suits, hearts, diamonds, spades, and clubs.

- **EXAMPLE 6 a)** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
- **b)** How many must be selected to guarantee that at least three hearts are selected?
- *Solution:* **a)** We know that at least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$.
- The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$,
- **b) I**n the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart.
- The next three cards will be all hearts, so we may need to select 42 cards to get three hearts.
- **EXAMPLE 7** A box contains 6 red balls , 9 green balls ,12 blue balls ,15 yellow balls , 20 white balls ,what is the minimum number of balls we must choose randomly from the box to ensure that we get at least 5 balls of same color .
- *Solution:* n =5 color ,k+1=5 balls , k=4
- Minimum number of ball require =kn+1 =4*5+1=21
- **EXAMPLE 8** What is minimum number of ball we must choose randomly from box to ensure that we get at least 10 balls of same color
- *Solution:* 6+9+9+9+9+1=43 balls
- **EXAMPLE 9** What is minimum number of ball we must choose randomly from box to ensure that we get at least 14 balls of same color
- *Solution:* 6+9+12+13+13+1=54 balls

- **Euler's totient function** counts the positive integers up to a given integer *n* that are relatively prime to *n*. It is written using the Greek letter phi as $\varphi(n)$ or Ø(n), and may also be called **Euler's phi function**.
- In other words, it is the number of integers *k* in the range $1 \le k \le n$ for which the greatest common divisor gcd(*n*, *k*) is equal to 1 .
- Co Prime of 6 ={1,5} , Ø(6)=2.
- Co Prime of 7 ={1,2,3,4,5,6} , Ø(7) = 6.
- If n is a prime number, then Ø(n) = n-1
- Else Ø(n) = $\frac{n(P_1-1)(P_2-1).......(P_k-1)}{P_1P_2P_3........P_k}$, for n = $P_1P_2P_3 \ldots \ldots P_k$
- **EXAMPLE 10** Number of +ve integers which are co prime to 110?
- *Solution:* 110=2*5*11
- Ø(110) = $\frac{110(2-1)(5-1)(11-1)}{2*5*11} = \frac{110*1*4*10}{2*5*11}$=40
- **EXAMPLE 11** Number of +ve integers which are co prime to 180?
- *Solution:* 180=2^2*3^2*5
- Ø(180) = $\frac{180(2-1)(3-1)(5-1)}{2*3*5} = \frac{180*1*2*4}{2*3*5}$=48

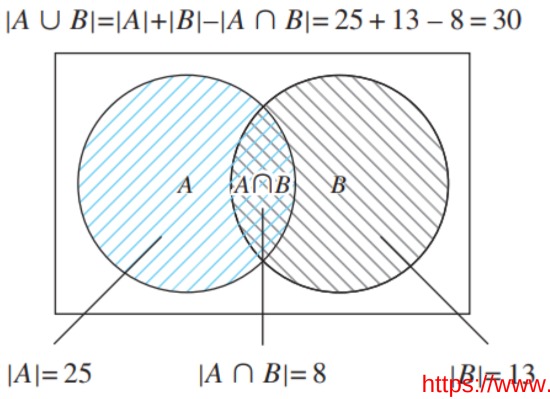
- **EXAMPLE 12** Number of +ve integers which are co prime to 323?
- *Solution:* 323=17*19
- $\emptyset(323) = \frac{323*(17-1)(19-1)}{17*19} = 288$
- **EXAMPLE 13** Let n=p^2*q ,Where p & q are prime number ,Number of int m ,s.t 1≤ m ≤ n and gcd of m and n=1 is ______ ?
- *Solution:* $\emptyset(n) = \frac{n*(p-1)(q-1)}{p*q}$ = p(p-1)(q-1)
- Number of Divisor / Factor of n =(a+1)(b+1)(c+1)(d+1)…..
- If n=p^a q^b r^c s^d…… Where p ,q ,r ,s are prime number and a , b , c, d are +ve int .
- **EXAMPLE 14** Number of divisors of 4200 is _________ ?
- *Solution:* 4200 = $2^3*3^1*5^2*7^1$
- Number of divisors=(3+1)(1+1)(2+1)(1+1)=4*2*3*2=48
- **EXAMPLE 15** Number of divisors of 2024 is _________ ?
- *Solution:* 2024 = $2^3*11^1*23^1$
- Number of divisors=(3+1)(1+1)(1+1)=4*2*2=16

- **Derangement** : A Permutation / Arrangement of n different objects in which no objects appear at it's correct place is called derangement .
- $D(n)=n!\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots\ldots \frac{(-1)^n}{n!}\}$
- $D(2)= 2!\{\frac{1}{2!}\}=1$
- $D(3)= 3!\{\frac{1}{2!} - \frac{1}{3!}\}= 6\{\frac{1}{2} - \frac{1}{6}\}= 6*\frac{2}{6} =2$
- $D(4)= 4!\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\}= 24\{\frac{1}{2} - \frac{1}{6}+\frac{1}{24}\}= 24*\frac{(12-4+1)}{24}=9$
- $D(5)= 5!\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\}= 120\{\frac{1}{2} - \frac{1}{6}+\frac{1}{24} - \frac{1}{120}\}= 120*\frac{(60-20+5-1)}{120} =44$
- **EXAMPLE 16** 5 different letters L_1,L_2,L_3,L_4,L_5 are to be placed in 5 envelopes e_1,e_2,e_3,e_4,e_5 one letter per envelope ,How many ways we can place the letter into envelope so that
- (I) No letter is correctly placed
- *Solution:* D(5)=44
- (II) At least one letter is correctly placed
- *Solution:* Number of way 5 letter put into 5 envelope =5!=120

- No letters in correct envelope =D(5)=44 ,So required number of way=120-44=76
- (III) Exactly two letters are correctly placed
- *Solution:* 2 letters from 5= ${}^{5}C_{2}$ =10 ,Remaining 3 letters can place in D(3)=2 way.
- So required number of way=10*2=20 way
- (IV) At most one letter is correctly placed
- *Solution:* 0 or 1 letter
- D(5)+C(5,1)*D(4)
- 44+5*9=44+45=89
- (V) At least one letter is wrongly placed
- *Solution:* There is only 1 way we can put all 5 letter correctly
- Rest all at least one will be wrongly placed
- 5!-1=120-1=119
- (VI) Exactly one letter is wrongly placed
- *Solution:* It is not possible to keep only one letter in wrong envelope

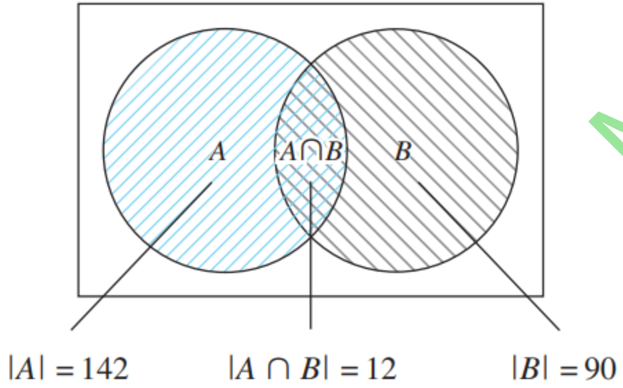
4.5 Inclusion–Exclusion

- **The Principle of Inclusion–Exclusion**
- The number of elements in the union of the two sets *A* and *B* is the sum of the numbers of elements in the sets minus the number of elements in their intersection. That is,
- $|A \cup B| = |A| + |B| - |A \cap B|$.
- **EXAMPLE 1** In a discrete mathematics class every student is a major in computer science or mathematics, or both. The number of students having computer science as a major is 25; the number of students having mathematics as a major is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in this class?
- *Solution:* Let *A* = set of students in the class majoring in computer science
- *B* = set of students in the class majoring in mathematics.
- $|A \cup B| = |A| + |B| - |A \cap B| = 25 + 13 - 8 = 30$.
- Therefore, there are 30 students in the class.

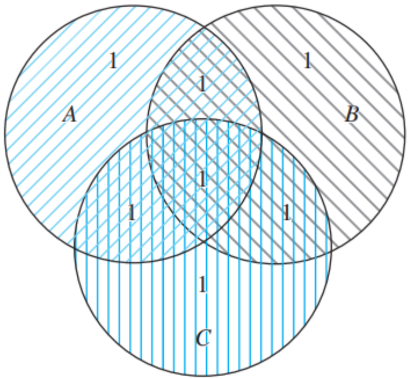


- **EXAMPLE 2** How many positive integers not exceeding 1000 are divisible by 7 or 11?
- *Solution:* Let A be the set of positive integers not exceeding 1000 that are divisible by 7, and let B be the set of positive integers not exceeding 1000 that are divisible by 11.
- $|A \cup B| = |A| + |B| - |A \cap B|$
- $\left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{7*11} \right\rfloor$
- $= 142 + 90 - 12 = 220$
- 220 positive integers not exceeding 1000 that are divisible by either 7 or 11.

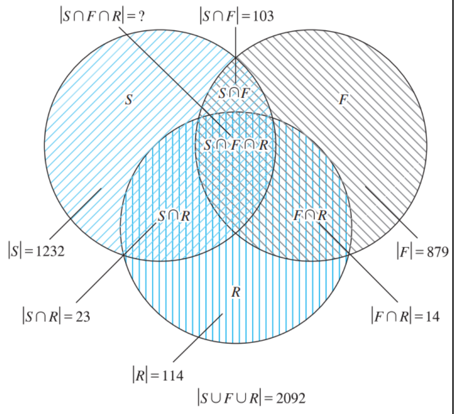
$|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220$



- **EXAMPLE 3** Suppose that there are 1807 freshmen at your school. Of these, 453 are taking a course in computer science, 567 are taking a course in mathematics, and 299 are taking courses in both computer science and mathematics. How many are not taking a course either in computer science or in mathematics?
- *Solution:* Let A be the set of all freshmen taking a course in computer science, and let B be the set of all freshmen taking a course in mathematics.
- $|A| = 453$, $|B| = 567$, and $|A \cap B| = 299$.
- The number of freshmen taking a course in either computer science or mathematics is
- $|A \cup B| = |A| + |B| - |A \cap B| = 453 + 567 - 299 = 721$.
- There are $1807 - 721 = 1086$ freshmen who are not taking a course in computer science or mathematics.
- **Finding a Formula for the Number of Elements in the Union of Three Sets**
- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.



- **EXAMPLE 4** A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?
- *Solution:* Let S be the set of students who have taken a course in Spanish,
- F the set of students who have taken a course in French,
- and R the set of students who have taken a course in Russian.
- Then $|S| = 1232$, $|F| = 879$, $|R| = 114$,
- $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$, and $|S \cup F \cup R| = 2092$.
- $|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$
- $2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$.
- $|S \cap F \cap R| = 7$.
- There are seven students who have taken courses in all.

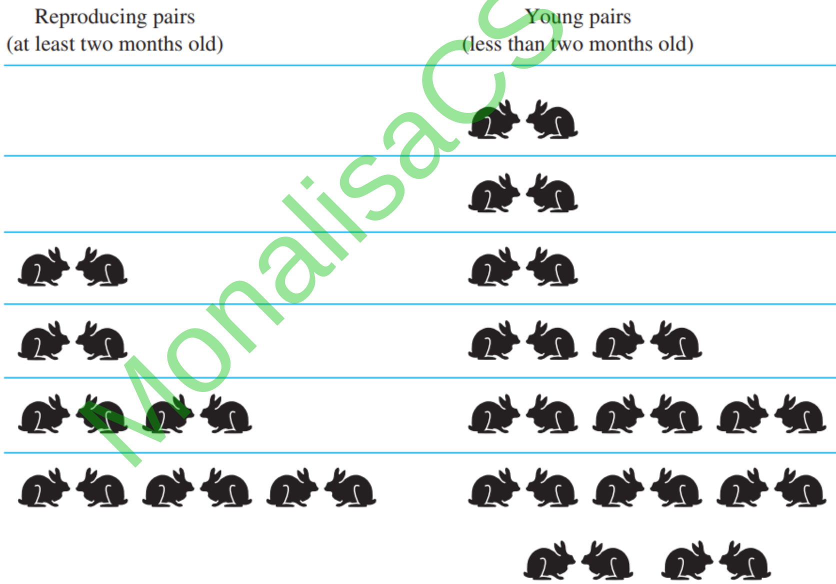


- **THEOREM 1** **THE PRINCIPLE OF INCLUSION–EXCLUSION** Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

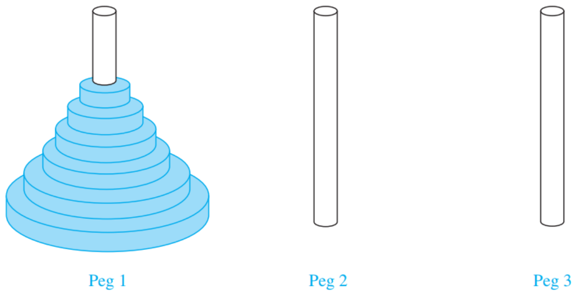
4.6 Recurrence Relations

- **EXAMPLE 1 Rabbits and the Fibonacci Numbers** Consider this problem, which was originally posed by Leonardo Pisano, also known as Fibonacci, in the thirteenth century in his book *Liber abaci*. A young pair of rabbits (one of each sex) is placed on an island.
- A pair of rabbits does not breed until they are 2 months old.
- After they are 2 months old, each pair of rabbits produces another pair each month, as shown in Figure.
- Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

- *Solution:* Denote by f_n the number of pairs of rabbits after n months. We will show that f_n, $n = 1, 2, 3, \ldots,$ are the terms of the Fibonacci sequence.
- $f_1 = 1, f_2 = 1$
- The sequence $\{f_n\}$ satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$
- The number of pairs of rabbits on the island after n months is given by the nth Fibonacci number.
- **EXAMPLE 2 The Tower of Hanoi** A popular puzzle of the late nineteenth century invented by the French mathematician Édouard Lucas, called the Tower of Hanoi, consists of three pegs mounted on a board together with disks of different sizes.
- Initially these disks are placed on the first peg in order of size, with the largest on the bottom .
- The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk.
- The goal of the puzzle is to have all the disks on the 3rd peg in order of size, with the largest on the bottom.



- *Solution:* Begin with n disks on peg 1. We can transfer the top n - 1 disks, following the rules of the puzzle, to 2nd peg using H_{n-1} moves.
- We keep the largest disk fixed during these moves.
- Then, we use one move to transfer the largest disk to the 3rd peg.
- We can transfer the n - 1 disks on peg 2 to peg 3 using H_{n-1} additional moves, placing them on top of the largest disk, which always stays fixed on the bottom of peg 3.
- This shows that $H_n = 2H_{n-1} + 1$.
- $H_n = 2H_{n-1} + 1$
- $= 2(2H_{n-2} + 1) + 1 = 2^2H_{n-2} + 2 + 1$
- $= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3H_{n-3} + 2^2 + 2 + 1$
- ...
- $= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \cdot\cdot\cdot + 2 + 1$
- $= 2^{n-1} + 2^{n-2} + \cdot\cdot\cdot + 2 + 1$
- $= 2^n - 1.$

- **EXAMPLE 3** Find a recurrence relation and give initial conditions for the number of bit strings of length *n* that do not have two consecutive 0s. How many such bit strings are there of length five?
- *Solution:* $a_1 = 2$, because both bit strings of length one, 0 and 1 do not have consecutive 0s,
- $a_2 = 3$, because the valid bit strings of length two are 01, 10, and 11.
- $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.
- $a_3 = a_2 + a_1 = 3 + 2 = 5$,
- $a_4 = a_3 + a_2 = 5 + 3 = 8$,
- $a_5 = a_4 + a_3 = 8 + 5 = 13$.
- **EXAMPLE 4** {a ,a+d ,a+2d ,a+3d ,…..} The recurrence relation is a $a_n = a_{n-1} + d(n \geq 1)$
- **EXAMPLE 5** {a ,ar,ar^2 ,ar^3,…..} The recurrence relation is a $a_n = a_{n-1}r\ (n \geq 1)$
- **DEFINITION 1** A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$,

  where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

- The recurrence relation in the definition is **linear** because the right-hand side is a sum of previous terms of the sequence each multiplied by a function of n.
- The recurrence relation is **homogeneous** because no terms occur that are not multiples of the a_js.
- The coefficients of the terms of the sequence are all **constants**, rather than functions that depend on n.
- The **degree** is k because a_n is expressed in terms of the previous k terms of the sequence.
- $a_0 = C_0, a_1 = C_1, . . . , a_{k-1} = C_{k-1}$.
- **EXAMPLE 6** The recurrence relation $P_n = (1.11)P_{n-1}$ is a linear homogeneous recurrence relation of degree one.
- The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two.
- The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five.
- **EXAMPLE 7** The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear.
- The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous.
- The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients.

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

- The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, where r is a constant.
- $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$ if and only if
- $r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_kr^{n-k}$.
- When both sides of this equation are divided by r^{n-k} and the right-hand side is subtracted from the left, we obtain the equation
- $r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_{k-1}r - c_k = 0$.
- The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution of this last equation.
- We call this the **characteristic equation** of the recurrence relation.
- The solutions of this equation are called the **characteristic roots** of the recurrence relation.
- **THEOREM 1** Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r_1^n + \alpha_2r_2^n$ for $n = 0, 1, 2, \ldots,$

  where α_1 and α_2 are constants.

- **EXAMPLE 8** What is the solution of the recurrence relation
- $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?
- *Solution:* The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$.
- $r^2 - 2r + r - 2 = 0$
- r(r-2)+1(r-2)=0
- (r - 2)(r + 1) = 0
- Its roots are $r = 2$ and $r = -1$.
- The solution to the recurrence relation is of the form, $a_n = \alpha_1 2^n + \alpha_2(-1)^n$.
- From the initial conditions, it follows that
- $a_0 = 2 = \alpha_1 + \alpha_2$,
- $a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$.
- Solving these two equations shows that $\alpha_1 = 3$ and $\alpha_2 = -1$.
- Hence, the solution to the recurrence relation $a_n = 3 \cdot 2^n - (-1)^n$.

- **EXAMPLE 9** Find an explicit formula for the Fibonacci numbers.
- *Solution:* $f_n = f_{n-1} + f_{n-2}$, initial conditions $f_0 = 0$ and $f_1 = 1$.
- The roots of the characteristic equation $r^2 - r - 1 = 0$ are
- $r = \frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A}$, A=1,B=-1,C=-1
- $r = \frac{1 \pm \sqrt{(1+4)}}{2} = \frac{1 \pm \sqrt{5}}{2}$
- $r_1 = (1 + \sqrt{5})/2$ and $r_2 = (1 - \sqrt{5})/2$.
- The solution to the recurrence relation is of the form, $f_n = \alpha_1 (\frac{1+\sqrt{5}}{2})^n + \alpha_2 (\frac{1-\sqrt{5}}{2})^n$
- From the initial conditions, it follows that
- $f_0 = 0 = \alpha_1 + \alpha_2$,
- $f_1 = 1 = \alpha_1 (\frac{1+\sqrt{5}}{2}) + \alpha_2 (\frac{1-\sqrt{5}}{2})$
- Solving these equations, we found $\alpha_1 = 1/\sqrt{5}$, $\alpha_2 = -1/\sqrt{5}$
- The formula for Fibonacci numbers are $f_n = \frac{1}{\sqrt{5}} (\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}} (\frac{1-\sqrt{5}}{2})^n$

- **THEOREM 2** Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots,$ where α_1 and α_2 are constants.
- **EXAMPLE 10** What is the solution of the recurrence relation
- $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$?
- *Solution:* The only root of $r^2 - 6r + 9 = 0$ is $r = 3$.
- Hence, the solution to this recurrence relation is $a_n = \alpha_1 3^n + \alpha_2 n 3^n$
- Using the initial conditions, it follows that
- $a_0 = 1 = \alpha_1,$
- $a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$
- Solving these two equations shows that $\alpha_1 = 1$ and $\alpha_2 = 1$.
- Consequently, the solution to this recurrence relation is
- $a_n = 3^n + n3^n$.

4.7 Generating Functions

- **DEFINITION 1** The *generating function for the sequence* $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series $G(x) = a_0 + a_1x + \cdots + a_kx^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$
- **EXAMPLE 1** The generating functions for the sequences $\{a_k\}$ with $a_k = 3$, $a_k = k + 1$, and $a_k = 2^k$ are $\sum_{k=0}^{\infty} 3x^k$, $\sum_{k=0}^{\infty}(k+1)x^k$, and $\sum_{k=0}^{\infty} 2^k x^k$, respectively.
- We can define generating functions for finite sequences of real numbers by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0$, $a_{n+2} = 0$, and so on.
- $G(x) = a_0 + a_1x + \cdots + a_nx^n$.
- **EXAMPLE 2** What is the generating function for the sequence 1, 1, 1, 1, 1, 1?
- *Solution:* The generating function of 1, 1, 1, 1, 1, 1 is
- $1 + x + x^2 + x^3 + x^4 + x^5$.
- By summation rule $\sum_{J=0}^{n} ar^j = \{\frac{ar^{n+1}-a}{r-1}$ if $r \neq 1$,$(n+1)a$ if $r=1\}$
- $a=1, r=x$
- $(x^6 - 1)/(x - 1) = 1 + x + x^2 + x^3 + x^4 + x^5$

- **EXAMPLE 3** Let m be a positive integer. Let $a_k = C(m, k)$, for $k = 0, 1, 2, \ldots, m$. What is the generating function for the sequence $a_0, a_1, \ldots, a_m$?
- *Solution:* The generating function $G(x) = C(m, 0) + C(m, 1)x + C(m, 2)x^2 + \cdots + C(m, m)x^m$.
- The binomial theorem shows that $G(x) = (1 + x)^m$.
- $(1\text{-}x)^m = C(m, 0) - C(m, 1)x + C(m, 2)x^2 + \cdots + (-1)^n C(m, m)x^m$.
- Sum of finite geometric sequence is , $a+ar^2+ar^3+\ldots ar^{n-1} = \frac{a(1-r^n)}{1-r}$ or $\frac{a(r^n-1)}{r-1}$,when $r \neq 1$
- Sum of infinite geometric sequence is , $a+ar^2+ar^3+\ldots = \frac{a}{1-r}$ when $r<1$
- **EXAMPLE 4** The function $f(x) = 1/(1 - x)$ is the generating function of the sequence 1, 1, 1, 1, . . . , because $1/(1 - x) = 1 + x + x^2 + \cdots$ for $|x| < 1$
- **EXAMPLE 5** The function $f(x) = 1/(1 - ax)$ is the generating function of the sequence 1, a, $a^2, a^3, \ldots$,because $1/(1 - ax) = 1 + ax + a^2x^2 + \ldots$ when $|ax| < 1$, or for $|x| < 1/|a|$ for $a \neq 0$
- **THEOREM 1** Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then
- $f(x) + g(x) = \sum_{k=0}^{\infty} (ak + bk)x^k$ and $f(x) * g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j}\right) x^k$.

- **DEFINITION 2** Let u be a real number and k a nonnegative integer. Then the *extended binomial coefficient* $\binom{u}{k}$ is defined by
- $\binom{u}{k} = \{\frac{(u)(u-1)......(u-k+1)}{k!}$ if k > 0 , 1 if k=0.
- **EXAMPLE 6** Find the values of the extended binomial coefficients $\binom{-2}{3}$ and $\binom{1/2}{3}$.
- *Solution:* u = -2 and k = 3 , $\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!}$ = -4.
- u = 1/2 and k = 3 , $\binom{1/2}{3} = \frac{(1/2)(1/2-1)(1/2-2)}{3!}$
- $= \frac{(1/2)(-1/2)(-3/2)}{3!}$ =1/16
- **EXAMPLE 7** When the top parameter is a negative integer, the extended binomial coefficient can be expressed in terms of an ordinary binomial coefficient.
- $\binom{-n}{r} = \frac{(-n)(-n-1)......(-n-r+1)}{r!}$ by definition of extended binomial coefficient
- $= \frac{(-1)^r(n)(n+1)......(n+r-1)}{r!}$ taking -1 common

- $=\frac{(-1)^r(n+r-1)(n+r-2)\ldots\ldots(n+1)(n)}{r!}$ by commutative law
- $=\frac{(-1)^r(n+r-1)!}{r!(n-1)!}$ multiplying both numerator and denominator by (n-1)!
- $\binom{-n}{r}$ = $(-1)^r$ C(n+r-1,r) or $(-1)^r \binom{n+r-1}{r}$
- **THEOREM 2 THE EXTENDED BINOMIAL THEOREM** Let *x* be a real number with $|x| < 1$ and let *u* be a real number. Then $(1+x)^u = \sum_{k=0}^{\infty}\binom{U}{k}x^k$
- **EXAMPLE 8** Find the generating functions for $(1 + x)^{-n}$ and $(1 - x)^{-n}$, where *n* is a positive integer, using the extended binomial theorem.
- *Solution:* By the extended binomial theorem, it follows that
- $(1+x)^{-n} = \sum_{k=0}^{\infty}\binom{-n}{k}x^k = \sum_{k=0}^{\infty}(-1)^k C(n+k-1,k)x^k$
- Replacing x by –x ,we find
- $(1-x)^{-n} = \sum_{k=0}^{\infty}\binom{-n}{k}(-x)^k = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n, k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k

$\dfrac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\dfrac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\dfrac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1, k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\dfrac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1, k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\dfrac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1, k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k} x^k = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

- **EXAMPLE 9** Solve the recurrence relation $a_k=3a_{k-1}$ for k = 1,2,3,... and initial condition a_0 = 2.
- *Solution:* Let $G(x)$ be the generating function for the sequence $\{a_k\}$, $G(x) = \sum_{k=0}^{\infty} a_k x^k$
- $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$
- Using the recurrence relation ,we see that
- $G(x) - 3xG(x) = \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1} x^k$
- $=a_0+\sum_{k=1}^{\infty}(ak - 3a_{k-1})x^k = 2$
- $G(x) - 3xG(x) = (1-3x)G(x) = 2$
- **EXAMPLE 10** Co-efficient of x^{20} in the sequence $(x^3+x^4+\ldots..)^5$ is __________
- *Solution:* $(x^3+x^4+\ldots..)^5 = x^{15}+x^{20}+x^{25}\ldots..$
- $= x^{15}(1+x^5+x^{10}\ldots..) = x^{15}(1+x+x^2\ldots..)^5 = x^{15}\{1/(1-x)\}^5 = x^{15}(1-x)^{-5}$
- $(1-x)^{-n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1+C(n,1)x + C(n+1,2)x^2+\ldots\ldots$
- $x^{15}(1-x)^{-5} = x^{15}\left(\sum_{k=0}^{\infty} C(5+k-1,k)x^k\right)$
- $= x^{15}\left(\sum_{k=0}^{\infty} C(4+k,k)x^k\right)$

- $= x^{15}(\sum_{k=0}^{\infty} C(4+k,k)x^k)$
- For k=5 , $x^{15} * C(9,5)x^5$
- $=126*x^{20}$, Co-efficient of x^{20} =126
- **EXAMPLE 11** Find the number of solutions of $e_1 + e_2 + e_3 = 17$,
- where e_1, e_2, and e_3 are nonnegative integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, and $4 \le e_3 \le 7$.
- *Solution:* The number of solutions with the indicated constraints is the coefficient of x^{17} in the expansion of
- $(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$.
- This follows because we obtain a term equal to x^{17} in the product by picking a term in the first sum x^{e_1}, a term in the second sum x^{e_2}, and a term in the third sum x^{e_3},
- Where the exponents e_1, e_2, and e_3 satisfy the equation $e_1 + e_2 + e_3 = 17$.
- $x^4 * x^6 * x^7 = x^{17}$, $x^5 * x^6 * x^6 = x^{17}$, $x^5 * x^5 * x^7 = x^{17}$
- The coefficient of x^{17} in this product is 3.
- Hence, there are three solutions.