Discrete Mathematics

Chapter 4 : Combinatorics

GATE CS PYQ

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Section1: Engineering Mathematics

- **Discrete Mathematics**: Propositional and first order logic. Sets, relations, functions, partial orders and lattices. Monoids, Groups. Graphs: connectivity, matching, coloring. Combinatorics: counting, recurrence relations , generating functions.
- **Linear Algebra**: Matrices, determinants, system of linear equations, eigenvalues and eigenvectors, LU decomposition.
- **Calculus**: Limits, continuity and differentiability. Maxima and minima. Mean value theorem. Integration.
- **Probability and Statistics**: Random variables. Uniform, normal, exponential, poisson and binomial distributions. Mean, median, mode and standard deviation. Conditional probability and Bayes theorem.

- **Discrete Mathematics**: Propositional and first order logic. Sets, relations, functions, partial orders and lattices.Monoids, Groups.Graphs: connectivity, matching, coloring.Combinatorics : counting, recurrence relations , generating functions.
- Chapter 1: Logic
- Propositional Logic, Propositional Equivalences , Predicates and Quantifiers , Nested Quantifiers , Rules of Inference , Introduction to Proofs.
- Chapter 2 : Set Theory
- Sets, relations, functions, partial orders and lattices.Monoids, Groups.
- Chapter 3 : Graph Theory
- Graphs: connectivity, matching, coloring.
- Chapter 4 : Combinatorics
- Counting, Recurrence relations , Generating functions

- **Chapter 4 : Combinatorics**
- 4.1 The Basics of Counting
- 4.2 Permutations and Combinations
- 4.3 Binomial Coefficients and Identities
- 4.4 The Pigeonhole Principle
- 4.5 Inclusion–Exclusion
- 4.6 Recurrence Relations
- 4.7 Generating Functions

- ***GATE CS 2014 Set 1 | Question: 37**
- There are 5 bags labeled 1 to 5. All the coins in a given bag have the same weight. Some bags have coins of weight 10 gm, others have coins of weight 11 gm. I pick 1,2,4,8,16 coins respectively from bags 1 to 5 Their total weight comes out to 323 gm. Then the product of the labels of the bags having 11 gm coins is ___.
- Let X = number of coins of 11 gm , Y = number of 10 gm coins.
- 11X+10Y=323
- X+Y=1+2+4+8+16=31
- 11X+10Y =323
- -10X-10Y =-310
- After solving X=13 ,Y=18
- Number of coins of 11 gm is 13
- All the coins in a given bag have the same weight either 11gm or 10 gm
- The only possible combination for 13 coins are
- 1(bag 1)+4(bag 3)+8(bag 4)=13
- Product of label of bags will be =1×3×4=12.
- Ans : 12

- ***GATE CS 2014 Set 1 | Question: 49**
- A pennant is a sequence of numbers, each number being 1 or 2. An n−pennant is a sequence of numbers with sum equal to n. For example, (1,1,2) is a 4−pennant. The set of all possible 1−pennants is (1), the set of all possible 2−pennants is (2),(1,1) and the set of all 3−pennants is (2,1),(1,1,1),(1,2). Note that the pennant (1,2) is not the same as the pennant (2,1). The number of 10−pennants is_________.
- Let f(n) = number of n pennants .
- f(1)=1 {(1)}
- f(2)=2 {(2),(1,1)}
- f(3)=3=2+1 {(2,1),(1,1,1),(1,2)}
- f(4)=5=3+2 {(2,2),(2,1,1),(1,2,1),(1,1,2),(1,1,1,1)}
- f(n)=f(n-1)+f(n-2)
- f(5)=8 ,f(6)=13 ,f(7)=21 ,f(8)=34 ,f(9)=55 ,
- f(10)=55+34=89
- Ans : 89

- ***GATE CS 2014 Set 2 | Question: 49**
- The number of distinct positive integral factors of 2014 is ______________
- Number of Divisor / Factor of n =(a+1)(b+1)(c+1)(d+1)…..
- If $n=p^a q^b r^c s^d$…… Where p ,q ,r ,s are prime number and a , b , c, d are +ve int
- $2014=2\times19\times53$
- Hence, total number of factors will be $=(1+1)\times(1+1)\times(1+1)=2\times2\times2=8$
- Ans : 8

- ***GATE CS 2015 Set 1 | Question: 49**
- Let a_n represent the number of bit strings of length n containing two consecutive 1s. What is the recurrence relation for a_n?
- A. $a_{n-2}+a_{n-1}+2^{n-2}$ B. $a_{n-2}+2a_{n-1}+2^{n-2}$ C. $2a_{n-2}+a_{n-1}+2^{n-2}$ D. $2a_{n-2}+2a_{n-1}+2^{n-2}$
- $a_1 = 0$, 0 and 1 do not have consecutive 1s,
- $a_2 = 1$, {11}
- $a_3 = 3$, {011,110,111}
- $a_4 = 8$, {0110,0011,0111,1011,1110,1100,1101,1111}
- A. a_3 =0+1+2=3 , a_4 = 1+3+4=8
- B. a_3 =0+2*1+2=4 , a_4 = 1+2*3+4=11
- C. a_3 =2*0+1+2=3 , a_4 = 2*1+3+4=9
- D. a_3 =2*0+2*1+2=4 , a_4 = 2*1+2*3+4=12
- Ans : A. $a_{n-2}+a_{n-1}+2^{n-2}$

- ***GATE CS 2015 Set 2 | Question: 9**
- The number of divisors of 2100 is ____.
- Number of Divisor / Factor of n =(a+1)(b+1)(c+1)(d+1)…..
- If $n=p^a q^b r^c s^d$…… Where p ,q ,r ,s are prime number and a , b , c, d are +ve int
- $2100=2^2\times3^1\times5^2\times7^1$
- Hence, total number of factors will be = $(2+1)\times(1+1)\times(2+1)\times(1+1)=3\times2\times3\times2=36$
- Ans : 36

- ***GATE CS 2015 Set 3 | Question: 5**
- The number of 4 digit numbers having their digits in non-decreasing order (from left to right) constructed by using the digits belonging to the set {1,2,3} is ________.
- 4 digit numbers having their digits in non-decreasing order (3333 , 2333 , 2233 , 2223 , 2222 , 1333 , 1233 , 1223 , 1222 , 1133 , 1123 , 1122 , 1113 ,1112 , 1111)
- 15 4 digit numbers
- Or ,
- cell (i,j) is starting with i and having j digits.
- We can have the relation c(i,j)=$\sum_{k=1}^{J}$ c(k,j−1)

	1 digit	**2 digits**	**3 digits**	**4 digits**
Starting 3	1	1	1	1
Starting 2	1	2	3	4
Starting 1	1	3	6	10

- 4 digit numbers=1+4+10=15
- Ans :15

- ***GATE CS 2016 Set 1 | Question: 2**
- Let a_n be the number of n-bit strings that do **NOT** contain two consecutive 1′s. Which one of the following is the recurrence relation for a_n?
- A. $a_n=a_{n-1}+2a_{n-2}$ B. $a_n=a_{n-1}+a_{n-2}$ C. $a_n=2a_{n-1}+a_{n-2}$ D. $a_n=2a_{n-1}+2a_{n-2}$
- $a_1 = 2$, because both bit strings of length one, 0 and 1 do not have consecutive 1s,
- $a_2 = 3$, because the valid bit strings of length two are 01, 10, and 00.
- $a_3 = 5$, because the valid bit strings of length three are 000,001, 010,100 and 101.
- $a_3 = a_2 + a_1 = 3 + 2 = 5$,
- $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.
- Ans : B. $a_n=a_{n-1}+a_{n-2}$

- ***GATE CS 2016 Set 1 | Question: 27**
- Consider the recurrence relation $a_1=8, a_n=6n^2+2n+a_{n-1}$. Let $a_{99}=K\times10^4$. The value of K is ___________.
- $a_n = 6n^2+2n+a_{n-1}$
- $= 6n^2+2n+(6(n-1)^2+2(n-1)+a_{n-2})$
- $= 6n^2+6(n-1)^2+2n+2(n-1)+(6(n-2)^2+2(n-2)+a_{n-3})$
- …….
- $=6(n^2+(n-1)^2+\ldots\ldots2^2)+2(n+(n-1)+\ldots.2)+a_1$
- $=6(\frac{n(n+1)(2n+1)}{6}-1)+2(\frac{n(n+1)}{2}-1)+8$
- $=n(n+1)(2n+1)-6+n(n+1)-2+8=n(n+1)(2n+2)$
- $=2n(n+1)^2$
- $a_{99}=2*99(99+1)^2=2*99(10)^4=198*(10)^4$
- K=198
- Ans: 198

- ***GATE CS 2017 Set 1 | Question: 47**
- The number of integers between 1 and 500 (both inclusive) that are divisible by 3 or 5 or 7 is ______________ .
- D(3) = 500/3=166
- D(5)=500/5=100
- D(7)=500/7=71
- D(3&5)=500/15=33
- D(3&7)=500/21=23
- D(5&7)=500/35=14
- D(3&5&7)=500/105=4
- D(3 or 5 or 7)= D(3)+D(5)+D(7)- D(3&5)-D(3&7)-D(5&7)+D(3&5&7)
- =166+100+71-33-23-14+4
- =341-70
- =271
- Ans: 271

- ***GATE CS 2017 Set 2 | Question: 47**
- If the ordinary generating function of a sequence $\{a_n\}_{n=0}^{\infty}$ is $\frac{1+z}{(1-z)^3}$, then $a_3 - a_0$ is equal to ____________ .
- $(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$
- $(1+z)(1-z)^{-3} = (1+z)\{ \sum_{k=0}^{\infty} C(3+k-1,k)z^k \}$
- $=(1+z)\{C(2,0)z^0+C(3,1)z^1+C(4,2)z^2+C(5,3)z^3+\ldots\ldots\}$
- $=(1+z)\{1+3z+6z^2+10z^3+\ldots\ldots\}$
- $=1+z+3z+3z^2+6z^2+6z^3+10z^3+10z^4+\ldots\ldots$
- $=1+4z+9z^2+16z^3+\ldots\ldots$
- a_0= coefficient of z^0 =1
- a_3= coefficient of z^3 =16
- $a_3 - a_0$ =16-1=15
- Ans :15

- ***GATE CS 2020 | Question: 42**
- The number of permutations of the characters in LILAC so that no character appears in its original position, if the two L's are indistinguishable, is ______.
- L I L A C
- _ _ _ _ _
- 1 2 3 4 5
- The two Ls cannot be placed at position 1 and 3 but they can be positioned at 2,4,5 in ${}^3C_2=3$ ways.
- Now one of 2,4,5 is vacant. lets say 2 is vacant.
- We can't palce I but we can place any of A, or C, so we have 2 choices for the position which is left after filling the two Ls. Now all of 2,4,5 are filled.
- For the remaining two places 1,3 we have two characters left and none of them is L so we can place them in 2!=2 ways.
- Total = $3*2*2!=3*2*2=12$.
- Ans :12

- ***GATE CS 2021 Set 1 | Question: 19**
- There are 6 jobs with distinct difficulty levels, and 3 computers with distinct processing speeds. Each job is assigned to a computer such that:
- The fastest computer gets the toughest job and the slowest computer gets the easiest job.
- Every computer gets at least one job.
- The number of ways in which this can be done is ___________.
- Let C_1,C_2,C_3 are 3 computers, C_1 is slowest and C_3 is fastest.
- Let jobs are J_1,J_2,J_3,J_4,J_5,J_6 with increasing difficulty level .
- $J_1 \rightarrow C_1$, $J_6 \rightarrow C_3$
- At least one of the jobs J_2,J_3,J_4,J_5 must be assigned to C_2.
- (J_2,J_3,J_4,J_5) can assign to (C_1,C_2,C_3) in $3^4=81$ ways
- (J_2,J_3,J_4,J_5) can assign to (C_1,C_3) in $2^4=16$ ways
- So required number of ways = 81-16=65
- Or , 1 jobs from (J_2,J_3,J_4,J_5) to C_2 rest 3 jobs to (C_1,C_3) $^4C_1*2^3=32$
- 2 jobs from (J_2,J_3,J_4,J_5) to C_2 rest 2 jobs to (C_1,C_3) $^4C_2*2^2=24$
- 3 jobs from (J_2,J_3,J_4,J_5) to C_2 rest 1 jobs to (C_1,C_3) $^4C_3*2=8$
- 4 jobs from (J_2,J_3,J_4,J_5) to C_2 rest 0 jobs to (C_1,C_3) $^4C_4=1$
- Total = 32+24+8+1=65 Ans : 65

- ***GATE CS 2022 | Question: 22**
- The number of arrangements of six identical balls in three identical bins is __________ .
- Given that, 3 Identical Bins and 6 Identical balls.
- Arrangements possible = (6,0,0),(5,1,0),(4,2,0),(4,1,1),(3,3,0),(3,2,1),(2,2,2)
- Only 7 arrangements are possible.
- Ans : 7

- ***GATE CS 2022 | Question: 26**

- Which one of the following is the closed form for the generating function of the sequence $\{a_n\}_{n\geq 0}$ defined below?
- a_n ={n+1,n is odd ={1,otherwise
- (A) $\frac{x(1+x^2)}{(1-x^2)^2} + \frac{1}{1-x}$ (B) $\frac{x(3-x^2)}{(1-x^2)^2} + \frac{1}{1-x}$ (C) $\frac{2x}{(1-x^2)^2} + \frac{1}{1-x}$ (D) $\frac{x}{(1-x^2)^2} + \frac{1}{1-x}$
- Generating function G(x) for the sequence a_n is $G(x) = \sum_{n=0}^{\infty} a_n x^n$
- The sequence $a_0x^0+a_1x^1+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+\ldots\ldots\infty$
- $=1+2x+x^2+4x^3+x^4+6x^5+x^6+8x^7+x^8+10x^9+x^{10}+\ldots\ldots\infty$
- $=(1+x^2+x^4+x^6+x^8+x^{10}\ldots\ldots\infty)+(2x+4x^3+6x^5+8x^7+10x^9+\ldots\ldots\infty)$
- $1+x^2+x^4+x^6+x^8\ldots\ldots\infty=\frac{1}{1-x^2}$ Let $2x+4x^3+6x^5+8x^7+10x^9+\ldots\ldots\infty=P$
- $Px^2= 2x^3+4x^5+6x^7+8x^9+10x^{11}+\ldots\ldots\infty$
- $P-Px^2=2x+2x^3+2x^5+2x^7+2x^9+2x^{11}+\ldots\ldots\infty$
- $=2x(1+x^2+x^4+x^6+x^8+x^{10}+\ldots\ldots\infty)$
- $P-Px^2=\frac{2x}{1-x^2}$

- $\Rightarrow P(1-x^2)=\dfrac{2x}{1-x^2} \Rightarrow P=\dfrac{2x}{(1-x^2)^2}$

(A) $\dfrac{x(1+x^2)}{(1-x^2)^2}+\dfrac{1}{1-x}$ (B) $\dfrac{x(3-x^2)}{(1-x^2)^2}+\dfrac{1}{1-x}$ (C) $\dfrac{2x}{(1-x^2)^2}+\dfrac{1}{1-x}$ (D) $\dfrac{x}{(1-x^2)^2}+\dfrac{1}{1-x}$

- $=\dfrac{1}{1-x^2}+\dfrac{2x}{(1-x^2)^2}$
- $=\dfrac{1+x-x}{1-x^2}+\dfrac{2x}{(1-x^2)^2}$
- $=\dfrac{1+x}{1-x^2}-\dfrac{x}{1-x^2}+\dfrac{2x}{(1-x^2)^2}$
- $=\dfrac{1+x}{(1-x)(1+x)}-\dfrac{x}{1-x^2}+\dfrac{2x}{(1-x^2)^2}$
- $=\dfrac{1}{(1-x)}-\dfrac{x}{1-x^2}+\dfrac{2x}{(1-x^2)^2}=\dfrac{1}{(1-x)}+\dfrac{2x}{(1-x^2)^2}-\dfrac{x}{1-x^2}$
- $=\dfrac{1}{(1-x)}+\dfrac{2x-x(1-x^2)}{(1-x^2)^2}$
- $=\dfrac{1}{(1-x)}+\dfrac{2x-x+x^3}{(1-x^2)^2}=\dfrac{1}{(1-x)}+\dfrac{x+x^3}{(1-x^2)^2}$
- Ans : (A) $\dfrac{x(1+x^2)}{(1-x^2)^2}+\dfrac{1}{1-x}$

- ***GATE CS 2023 | Question: 5**
- The Lucas sequence L_n is defined by the recurrence relation : $L_n=L_{n-1}+L_{n-2}$, for $n\geq3$, with $L_1=1$ and $L_2=3$. Which one of the options given is TRUE?
- A. $L_n= (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n$ B. $L_n= (\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{3})^n$
- C. $L_n= (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{3})^n$ D. $L_n= (\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n$
- $L_n=L_{n-1}+L_{n-2}$, initial conditions $L_1=1$ and $L_2=3$.
- The roots of the characteristic equation $r^2 - r - 1 = 0$ are
- $r = \frac{-B\pm\sqrt{(B^2-4AC)}}{2A}$, A=1,B=-1,C=-1 $r = \frac{1\pm\sqrt{(1+4)}}{2} = \frac{1\pm\sqrt{5}}{2}$, $r_1 = (1 + \sqrt{5})/2$ and $r_2 = (1 - \sqrt{5})/2$.
- The solution to the recurrence relation is of the form, $L_n = \alpha_1(\frac{1+\sqrt{5}}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n$
- From the initial conditions, it follows that
- $L_1=1 = \alpha_1(\frac{1+\sqrt{5}}{2}) + \alpha_2(\frac{1-\sqrt{5}}{2})$ $L_2=3 = \alpha_1(\frac{1+\sqrt{5}}{2})^2 + \alpha_2(\frac{1-\sqrt{5}}{2})^2$
- If $\alpha_1 =1$, $\alpha_2 =1$ then both initial condition satisfy
- The formula for Lucas sequence $L_n = (\frac{1+\sqrt{5}}{2})^n +(\frac{1-\sqrt{5}}{2})^n$
- Ans : A. $L_n= (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n$

- ***GATE CS 2023 | Question: 38**
- Let U={1,2,…,n}, where n is a large positive integer greater than 1000. Let k be a positive integer less than n. Let A,B be subsets of U with |A|=|B|=k and A∩B=∅. We say that a permutation of U separates A from B if one of the following is true.
- All members of A appear in the permutation before any of the members of B.
- All members of B appear in the permutation before any of the members of A.
- How many permutations of U separate A from B?
- (A) n! (B) $\binom{n}{2k}$(n−2k) ! (C) $\binom{n}{2k}$(n−2k)!$(k!)^2$ (D)2 $\binom{n}{2k}$(n−2k)!$(k!)^2$
- |A|=|B|=k
- A can appear before B or B can appear before A=2way
- Select 2k integers from n integers for A and B = $\binom{n}{2k}$
- Elements in A and B can permute amongst themselves so k!∗k! = $(k!)^2$
- The remaining elements (n-2k) can arrange in any way so = (n−2k)!
- Permutations of U separate A from B = 2 $\binom{n}{2k}$(n−2k)!$(k!)^2$
- Ans : (D)2 $\binom{n}{2k}$(n−2k)!$(k!)^2$