Algorithms

Chapter 7: Dynamic Programming

GATE CS PYQ

Solved by Monalisa

Section 5: Algorithms

- Searching, sorting, hashing. Asymptotic worst case time and space complexity. Algorithm design techniques : greedy, dynamic programming and divide-and-conquer . Graph traversals, minimum spanning trees, shortest paths
- Chapter 1:Algorithim Analysis:-Algorithm intro , Order of growth ,Asymptotic notation, Time complexity, space complexity, Analysis of Recursive & non recursive program, Master theorem]
- Chapter 2:Brute Force:-Sequential search, Selection Sort and Bubble Sort , Radix sort, Depth first Search and Breadth First Search.
- Chapter 3: Decrease and Conquer :- Insertion Sort, Topological sort,Binary Search .
- Chapter 4: Divide and conquer:-Min max problem , matrix multiplication ,Merge sort ,Quick Sort , Binary Tree Traversals and Related Properties .
- Chapter 5: Transform and conquer:- Heaps and Heap sort, Balanced Search Trees.
- Chapter 6: Greedy Method:-knapsack problem , Job Assignment problem, Optimal merge, Hoffman Coding, minimum spanning trees, Dijkstra's Algorithm.
- Chapter 7: Dynamic Programming:-The Bellman-Ford algorithm ,Warshall's and Floyd's Algorithm ,Rod cutting, Matrix-chain multiplication ,Longest common subsequence ,Optimal binary search trees
- Chapter 8: Hashing.
- Reference : Introduction to Algorithms by Thomas H. Cormen
- Introduction to the Design and Analysis of Algorithms, by Anany Levitin
- My Note

- Chapter 7: Dynamic Programming:-
- Bellman-Ford single-source shortest path
- **GATE CS 2013 | Question: 19**
- The Floyd-Warshall algorithm,
- **GATE CSE 2016 Set 2 | Question: 14 , GATE CSE 2021 Set 1 | Question: 36**
- Matrix chain product,
- **GATE CS 2018 | Question: 31, GATE CSE 2011 | Question: 38 , GATE CS 2016 Set 2 | Question: 38**
- Longest common subsequence
- **GATE CSE 2011 | Question: 25, GATE CS 2014 Set 2 | Question: 37**
- Optimal Binary Search trees.

- **GATE CS 2009 | Question: 13**
- Which of the following statement(s) is/are correct regarding Bellman-Ford shortest path algorithm?
- **P:** Always finds a negative weighted cycle, if one exists.
- **Q:** Finds whether any negative weighted cycle is reachable from the source.
- (A)P only (B)Q only (C)Both P and Q (D)Neither P nor Q
- The Bellman-Ford algorithm returns a boolean value indicating whether or not there is a negative-weight cycle that is reachable from the source.
- P:False
- Q:True
- Ans : (B)Q only

- **GATE CS 2011 | Question: 25**
- An algorithm to find the length of the longest monotonically increasing sequence of numbers in an array A[0:n−1] is given below.
- Let L_i, denote the length of the longest monotonically increasing sequence starting at index i in the array.
- Initialize $L_{n-1}=1$. For all i such that $0 \leq i \leq n-2$
- $L_i =$ {$1+L_{i+1}$ if $A[i] < A[i+1]$
- $=$ {1 Otherwise
- Finally, the length of the longest monotonically increasing sequence is $\max(L_0, L_1, \ldots, L_{n-1})$.
- Which of the following statements is **TRUE**?
- A.The algorithm uses dynamic programming paradigm
- B.The algorithm has a linear complexity and uses branch and bound paradigm
- C.The algorithm has a non-linear polynomial complexity and uses branch and bound paradigm
- D.The algorithm uses divide and conquer paradigm
- The algorithm is storing the optimal solutions to subproblems at each i, and using it to derive the optimal solution of a bigger problem.
- This is dynamic programming approach.
- Ans : A.The algorithm uses dynamic programming paradigm

GATE CS 2011 | Question: 38

- Four Matrices M_1,M_2,M_3 and M_4 of dimensions p×q,q×r,r×s and s×t respectively can be multiplied in several ways with different number of total scalar multiplications. For example when multiplied as $((M_1 \times M_2) \times (M_3 \times M_4))$, the total number of scalar multiplications is pqr+rst+prt. When multiplied as $(((M_1 \times M_2) \times M_3) \times M_4)$, the total number of scalar multiplications is pqr+prs+pst.
- If p=10,q=100,r=20,s=5 and t=80, then the minimum number of scalar multiplications needed is
- A. 248000 B. 44000 C. 19000 D.25000

Matrix	M_1	M_2	M_3	M_4	p_0	p_1	p_2	p_3	p_4
Dimension	10×100	100×20	20×5	5×80	10	100	20	5	80

- **$m[i, j] = 0$ if $i=j$, $\min\limits_{i \le k < j} \{m[i,k]+ m[k+1,j]+p_{i-1}p_kp_j\}$ if $i<j$.**
- m[1,2]=10*100*20=20000,m[2,3]=100*20*5=10000,m[3,4]=20*5*80=8000
- m[1,3]=Min {k=1: m[1,1]+m[2,3] + $p_0 p_1 p_3$ =0+10000+10*100*5
- =15000,k=2: m[1,2]+m[3,3]+$p_0 p_2 p_3$=20000+0+10*20*5=21000}
- m[2,4]=Min{k=2: m[2,2]+m[3,4] + $p_1 p_2 p_4$= 0+8000+100*20*80=168000,
- k=3: m[2,3]+m[4,4]+$p_1 p_3 p_4$=10000 +0+100*5*80 =50000}
- m[1,4]=Min{k=1: m[1,1]+m[2,4] + $p_0 p_1 p_4$= 0+50000+10*100*80
- =130000,k=2: m[1,2]+m[3,4]+$p_0 p_2 p_4$=20000+8000+10*20*80=
- 44000 ,k=3: m[1,3]+m[4,4] + $p_0 p_3 p_4$=15000+0+10*5*80=19000}
- $((M_1(M_2\ M_3))\ M_4)$=19000 ,Ans : C. 19000

j

1	2	3	4	m
0	20000	15000	19000	1
	0	10000	50000	2
		0	8000	3
			0	4

i

j

2	3	4	s
1	1	3	1
	2	3	2
		3	3

i

- **GATE CS 2013 | Question: 19**
- What is the time complexity of Bellman-Ford single-source shortest path algorithm on a complete graph of n vertices?
- A. $\Theta(n^2)$ B. $\Theta(n^2 \log n)$ C. $\Theta(n^3)$ D. $\Theta(n^3 \log n)$
- Time complexity of Bellman-Ford algorithm is $\Theta(|V||E|)$.
- $|V|=n$
- If the graph is complete, the value of $|E| =|V|^2=n^2$
- Time complexity = $\Theta(n^3)$
- Ans : C. $\Theta(n^3)$

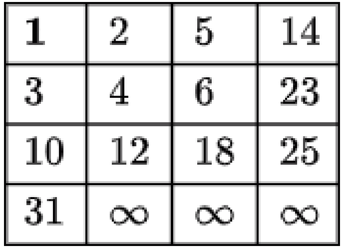
GATE CS 2014 Set 2 | Question: 37

- Consider two strings A=*"qpqrr"* and B=*"pqprqrp"*. Let x be the length of the longest common subsequence *(not necessarily contiguous)* between A and B and let y be the number of such longest common subsequences between A and B. Then x+10y= ___.
- $c[i,j]=$ $\{0$ if $i=0$ or $j=0$,
- $\{c[i-1,j-1]+1$ if $i,j>0$ and $x_i=y_j$,
- $\{\max(c[i,j-1],c[i-1,j])$ if $i,j>0$ and $x_i\neq y_j$.

	j	0	1	2	3	4	5	6	7
i		y_j	p	q	p	r	q	r	p
0	x_i	0	0	0	0	0	0	0	0
1	q	0	↑ 0	↖ 1	← 1	← 1	↖ 1	← 1	← 1
2	p	0	↖ 1	↑ 1	↖ 2	← 2	← 2	← 2	↖ 2
3	q	0	↑ 1	↖ 2	↑ 2	↑ 2	↖ 3	← 3	← 3
4	r	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4	← 4
5	r	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4	↑ 4

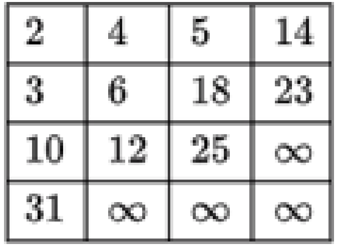
- x=4
- LCS are qpqr , pqrr ,qprr
- y=3
- x+10y=34
- Ans :34

- **GATE CS 2015 Set 2 | Question: 31**
- A Young tableau is a 2D array of integers increasing from left to right and from top to bottom. Any unfilled entries are marked with ∞, and hence there cannot be any entry to the right of, or below a ∞. The following Young tableau consists of unique entries.
- When an element is removed from a Young tableau, other elements should be moved into its place so that the resulting table is still a Young tableau (unfilled entries may be filled with a ∞). The minimum number of entries (other than 1) to be shifted, to remove 1 from the given Young tableau is ______.



1	2	5	14
3	4	6	23
10	12	18	25
31	∞	∞	∞

- 1: shift 2 left in place of 1
- 2: 4 is shifted up to the first row
- 3: 6 is shifted left in second row .
- 4: 18 shifts up to the second row
- 5: 25 is shifted left to the third column.
- Also infinity is placed to the right of 25 and below 23.
- So, 5 moves and still maintains the tableau property.
- Ans : 5



2	4	5	14
3	6	18	23
10	12	25	∞
31	∞	∞	∞

- **GATE CS 2016 Set 2 | Question: 14**
- The Floyd-Warshall algorithm for all-pair shortest paths computation is based on
- A. Greedy paradigm.
- B. Divide-and-conquer paradigm.
- C. Dynamic Programming paradigm.
- D. Neither Greedy nor Divide-and-Conquer nor Dynamic Programming paradigm
- In Floyd Warshall's, we calculate all possibilities and select best one
- So its Dynamic Programming Paradigm.
- Ans: C. Dynamic Programming paradigm.

GATE CS 2016 Set 2 | Question: 38

- Let A_1,A_2,A_3 and A_4 be four matrices of dimensions 10×5,5×20,20×10 and 10×5, respectively. The minimum number of scalar multiplications required to find the product $A_1A_2A_3A_4$ using the basic matrix multiplication method is __________.

Matrix	A_1	A_2	A_3	A_4	p_0	p_1	p_2	p_3	p_4
Dimension	10×5	5×20	20×10	10×5	10	5	20	10	5

- **$m[i, j]$ =0 if i=j, $\min\limits_{i\le k<j}$ {$m[i,k]+ m[k+1,j]+p_{i-1}p_kp_j$} if i<j.**
- m[1,2]=10*5*20=1000,m[2,3]=5*20*10=1000,m[3,4]=20*10*5=1000
- m[1,3]=Min {k=1: m[1,1]+m[2,3] + $p_0p_1p_3$ =0+1000+10*5*10 =1500,
- k=2: m[1,2]+m[3,3]+$p_0p_2p_3$=1000+0+10*20*10=3000}
- m[2,4]=Min{k=2: m[2,2]+m[3,4] + $p_1p_2p_4$= 0+1000+5*20*5=1500,
- k=3: m[2,3]+m[4,4]+$p_1p_3p_4$=1000 +0+5*10*5 =1250}
- m[1,4]=Min{k=1: m[1,1]+m[2,4] + $p_0p_1p_4$= 0+1250+10*5*5=1500,
- k=2: m[1,2]+m[3,4]+$p_0p_2p_4$=1000+1000+10*20*5=3000 ,
- k=3: m[1,3]+m[4,4] + $p_0p_3p_4$=1500+0+10*10*5=2000}
- $(A_1((A_2\ A_3)\ A_4))$=1500 ,
- Ans : 1500

j

1	2	3	4	m	
0	1000	1500	1500	1	
	0	1000	1250	2	i
		0	1000	3	
			0	4	

j

2	3	4	s	
1	1	1	1	
	2	3	2	i
		3	3	

- **GATE CS 2018 | Question: 31**
- Assume that multiplying a matrix G_1 of dimension $p \times q$ with another matrix G_2 of dimension $q \times r$ requires pqr scalar multiplications. Computing the product of n matrices $G_1 G_2 G_3 \ldots G_n$ can be done by parenthesizing in different ways. Define $G_i G_{i+1}$ as an **explicitly computed pair** for a given parenthesization if they are directly multiplied. For example, in the matrix multiplication chain $G_1 G_2 G_3 G_4 G_5 G_6$ using parenthesization $(G_1(G_2G_3))(G_4(G_5G_6))$, G_2G_3 and G_5G_6 are only explicitly computed pairs.
- Consider a matrix multiplication chain $F_1F_2F_3F_4F_5$, where matrices F_1, F_2, F_3, F_4 and F_5 are of dimensions 2×25,25×3,3×16,16×1 and 1×1000, respectively. In the parenthesization of $F_1F_2F_3F_4F_5$ that minimizes the total number of scalar multiplications, the explicitly computed pairs is/are
- A.F_1F_2 and F_3F_4 only B.F_2F_3 only C.F_3F_4 only D.F_1F_2 and F_4F_5 only
- Matrix F_5 is of dimension 1 × 1000, which will increase multiplication cost.
- So evaluating F_5 at last is optimal.
- $((F_1(F_2(F_3\ F_4)))F_5)$: 48 + 75 + 50 + 2000 = 2173
- F_3, F_4 are explicitly computed pairs.
- Ans: C.F_3F_4 only

Matrix	F_1	F_2	F_3	F_4	p_0	p_1	p_2	p_3	p_4
Dimension	2×25	25×3	3×16	16 ×1	2	25	3	16	1

- $m[i, j] = 0$ if $i=j$, $\min\limits_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1} p_k p_j\}$ if $i<j$.
- m[1,2]=2*25*3=150,m[2,3]=25*3*16=1200,m[3,4]=3*16*1=48
- m[1,3]=Min {k=1: $m[1,1]+m[2,3] + p_0 p_1 p_3$ =0+1200+2*25*16 =2000,
- k=2: $m[1,2]+m[3,3]+p_0 p_2 p_3$=150+0+2*3*16=246}
- m[2,4]=Min{k=2: $m[2,2]+m[3,4] + p_1 p_2 p_4$= 0+48+25*3*1=123,
- k=3: $m[2,3]+m[4,4]+p_1 p_3 p_4$=1200 +0+25*16*1 =1600}
- m[1,4]=Min{k=1: $m[1,1]+m[2,4] + p_0 p_1 p_4$= 0+123+2*25*1=173,
- k=2: $m[1,2]+m[3,4]+p_0 p_2 p_4$=150+48+2*3*1=204 ,
- k=3: $m[1,3]+m[4,4] + p_0 p_3 p_4$=246+0+2*16*1=278}
- $(F_1(F_2\ (F_3\ F_4)))$=173
- Now with F_5
- $((F_1(F_2\ (F_3\ F_4)))\ F_5)$ =173+2*1*1000=2173

j

1	2	3	4	m
0	150	246	173	1
	0	1200	123	2
		0	48	3
			0	4

i

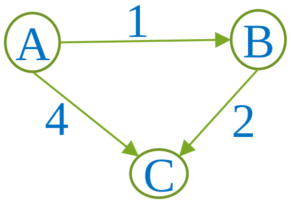
j

2	3	4	s
1	2	1	1
	2	2	2
		3	3

i

GATE CS 2020 | Question: 40

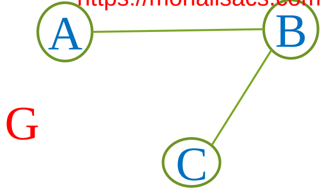
- Let $G=(V,E)$ be a directed, weighted graph with weight function $w: E \rightarrow R$. For some function $f: V \rightarrow R$, for each edge$(u,v) \in E$, define $w'(u,v)$ as $w(u,v)+f(u)-f(v)$.
- Which one of the options completes the following sentence so that it is TRUE?



- “The shortest paths in G under w are shortest paths under w' too,________________”.
- (A)for every $f: V \rightarrow R$ (B)if and only if $\forall u \in V, f(u)$ is positive
- (C)if and only if $\forall u \in V, f(u)$ is negative
- (D) if and only if $f(u)$ is the distance from s to u in the graph obtained by adding a new vertex s to G and edges of zero weight from s to every vertex of G.
- w(A,C)=3
- w'(A,C)=w(A,C)+f(A)-f(C) =w(A,C)+constant ,if f(A) & f(C)=constant
- w'(u,v) is a transformation of w(u,v).
- (A) true for every f:V→R
- (B,C) $f(u)$ positive or negative will not affect .
- (D) $f(u)-f(v)=0$ as zero weight from s to every vertex of G.
- This will be true if deference is either constant or 0.
- In option D its written if and only if so false.
- Ans : (A)for every $f: V \rightarrow R$

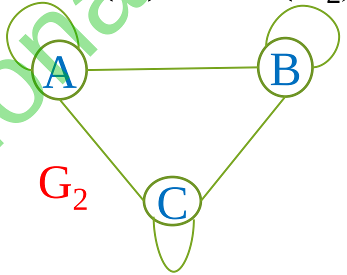
GATE CS 2021 Set 1 | Question: 36

- Let G=(V,E) be an undirected unweighted connected graph. The *diameter* of G is defined as: diam(G)=$\max_{u,v\epsilon V}$ {the length of shortest path between *u* and *v*}
- Let M be the adjacency matrix of G.
- Define graph G_2 on the same set of vertices with adjacency matrix N, where
- N_{ij}={1 if M_{ij}>0 or P_{ij}>0, where $P=M^2$
- ={0 otherwise
- Which one of the following statements is true?
- A.diam(G_2)≤[diam(G)/2] B.[diam(G)/2]<diam(G_2)<diam(G)
- C.diam(G_2)=diam(G) D.diam(G)<diam(G_2)≤2diam(G)
- diam(G)=max{1,1,2}=2
- diam(G_2)=max{1,1,1}=1
- A.1≤ 2/2 True
- B.1<1<2 False
- C.1=2 False
- D.2<1 ≤2*2 False
- Ans: A.diam(G_2)≤[diam(G)/2]



$$\mathbf{M}\quad \begin{array}{c|ccc|} & A & B & C \\ A & 0 & 1 & 0 \\ B & 1 & 0 & 1 \\ C & 0 & 1 & 0 \end{array}$$

$$\mathbf{M^2}\ P\quad \begin{array}{c|ccc|} & A & B & C \\ A & 1 & 0 & 1 \\ B & 0 & 2 & 0 \\ C & 1 & 0 & 1 \end{array}$$



$$\mathbf{N}\quad \begin{array}{c|ccc|} & A & B & C \\ A & 1 & 1 & 1 \\ B & 1 & 1 & 1 \\ C & 1 & 1 & 1 \end{array}$$

- **GATE CS 2021 Set 1 | Question: 40**
- Define $\boldsymbol{R_n}$ to be the maximum amount earned by cutting a rod of length n meters into one or more pieces of integer length and selling them. For *i*>0, let p[*i*] denote the selling price of a rod whose length is *i* meters. Consider the array of prices:
- p[1]=1,p[2]=5,p[3]=8,p[4]=9,p[5]=10,p[6]=17,p[7]=18Which of the following statements is/are correct about $\boldsymbol{R_7}$?
- A. $\boldsymbol{R_7}$=18 B. $\boldsymbol{R_7}$=19
- C. $\boldsymbol{R_7}$ is achieved by three different solutions
- D. $\boldsymbol{R_7}$ cannot be achieved by a solution consisting of three pieces
- $R_n = \max_{1 \le i \le n} (p_i + R_{n-i})$
- $R_1=1$, 1=1 no cuts $R_2=5$, 2=2 no cuts $R_3=8$, 3=3 no cuts
- $R_4=10$, 4=2+2 $R_5=13$, 5=2+3 $R_6=17$, 6=6 no cuts
- $R_7=\max\{p_1+R_6, p_2+R_5 , p_3+R_4 , p_4+R_3 , p_5+R_2 , p_6+R_1 , p_7+R_0 \}$
- $R_7=\max\{1+17, 5+13 , 8+10 , 9+8 , 10+5 , 17+1 , 18 \}=18$
- 7,1+6,2+2+3 Three different solutions.
- Ans : A,C