Discrete Mathematics
Chapter 1: Propositional and first order logic

GATE CS Lectures
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Section1: Engineering Mathematics

- **Discrete Mathematics**: Propositional and first order logic. Sets, relations, functions, partial orders and lattices. Monoids, Groups. Graphs: connectivity, matching, coloring. Combinatorics: counting, recurrence relations , generating functions.
- **Linear Algebra**: Matrices, determinants, system of linear equations, eigenvalues and eigenvectors, LU decomposition.
- **Calculus**: Limits, continuity and differentiability. Maxima and minima. Mean value theorem. Integration.
- **Probability and Statistics**: Random variables. Uniform, normal, exponential, poisson and binomial distributions. Mean, median, mode and standard deviation. Conditional probability and Bayes theorem.

- **Discrete Mathematics**: Propositional and first order logic. Sets, relations, functions, partial orders and lattices.Monoids, Groups.Graphs: connectivity, matching, coloring.Combinatorics : counting, recurrence relations , generating functions.
- Chapter 1: Logic
- Propositional Logic, Propositional Equivalences , Predicates and Quantifiers , Nested Quantifiers , Rules of Inference , Introduction to Proofs.
- Chapter 2 : Set Theory
- Chapter 3 : Graph Theory
- Chapter 4 : Combinatorics

- **Ch 1:Mathematical Logic:**
- 1.1 Propositional Logic,
- 1.2 Propositional Equivalences,
- 1.3 Predicates and Quantifiers,
- 1.4 Nested Quantifiers,
- 1.5 Rules of Inference,
- 1.6 Introduction to Proofs.

1.1 Propositional Logic

- **Introduction**
- Logic is the basis of all mathematical reasoning, and has numerous applications to computer science.
- It used in the design of computer circuits ,construction of computer programs, verification of the correctness of programs, design of computing machines, artificial intelligence.
- **Propositions**
- A **proposition** is a declarative sentence (a sentence that declares a fact) that is either true or false, but not both. A proposition is the basic building block of logic.
- **EXAMPLE 1** All the following declarative sentences are propositions.
- 1: Chicago is one of the largest cities in the U.S. 2:1 + 1 = 2. 3:2 + 2 = 3.
- Propositions 1 and 2 are true, whereas 3 is false
- **EXAMPLE 2** Consider the following sentences.
- 1. What time is it? 2. Read this carefully. 3. $x + 1 = 2$. 4. $x + y = z$.
- Sentences 1 and 2 are not propositions because they are not declarative sentences.
- Sentences 3 and 4 are not propositions because they are neither true nor false.
- But sentences 3 and 4 can be turned into a proposition.

- We use letters to denote **propositional variables** (or **statement variables**), just as letters are used *p*, *q*, *r*, *s*, ...
- The **truth value** of a proposition is true, denoted by **T**, a true proposition, and the truth value of a proposition is false, denoted by **F**, if it is a false proposition.
- **Propositional calculus** or **propositional logic** deals with propositions.
- Many mathematical statements are constructed by combining one or more propositions.
- New propositions, called **compound propositions**, are formed from existing propositions using logical operators.
- **A** proposition which can't be divide further into two or more proposition is said to be Atomic denoted by propositional variables p,q,r,s…
- **LOGICAL OPERATORS :**
- **DEFINITION 1** Let p be a proposition. The ***negation** of* p, denoted by $\neg p$ (also $\bar{p}$), is the statement "It is not the case that p." The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.
- **EXAMPLE 3** p: It is raining today
- ~p :It is not raining today
- or It is not the case that it is raining today.
- The logical operators that are used to form new propositions from two or more existing propositions called **connectives.**

TABLE 1 The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

- **DEFINITION 2** Let p and q be propositions. The ***conjunction*** of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.
- Note that in logic the word "but" sometimes is used instead of "and" in a conjunction.
- For example, the statement "The sun is shining, but it is raining" is another way of saying "The sun is shining and it is raining."
- **EXAMPLE 4** p : "Rebecca's PC has more than 64 GB free hard disk space" and q : "The processor in Rebecca's PC runs faster than 1 GHz."
- $p \wedge q$: "Rebecca's PC has more than 64 GB free hard disk space, and its processor runs faster than 1 GHz.".
- **DEFINITION 3** Let p and q be propositions. The ***disjunction*** of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.
- *or* can be used in two ways **inclusive or**
- "Students who have taken calculus or computer science can take this class."

TABLE 2 **The Truth Table for the Conjunction of Two Propositions.**

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- Here, we mean that students who have taken both calculus and computer science can take the class, as well as the students who have taken only one of the two subjects.
- **exclusive or**
- "Students who have taken calculus or computer science, but not both, can enroll in this class."
- Here, we mean that students who have taken both calculus and a computer science course cannot take the class. Only those who have taken exactly one of the two can take the class.
- **DEFINITION 4** Let p and q be propositions. The ***exclusive or*** of p and q, denoted by p ⊕ q, is the proposition that is true when exactly one of p and q is true and is false otherwise.

TABLE 3 The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- **DEFINITION 5** Let p and q be propositions. The ***conditional statement*** $p \rightarrow q$ is the proposition "if p, then q." The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).
- The statement $p \rightarrow q$ is called a conditional statement because $p \rightarrow q$ asserts that q is true on the condition that p holds. A conditional statement is also called an **implication**.
- A variety of terminology is used to express $p \rightarrow q$.
- "if p, then q" "if p, q" "p only if q" "a necessary condition for p is q"
- "q if p" "q when p" "q unless $\neg p$" "a sufficient condition for q is p"
- "p implies q" "q follows from p" "q whenever p" "q is necessary for p"
- "p is sufficient for q"
- "If I am elected, then I will lower taxes."
- **EXAMPLE 5** p : "Maria learns discrete mathematics" and q : "Maria will qualify GATE."
- $p \rightarrow q$: "If Maria learns discrete mathematics, then she will qualify GATE." or "Maria will qualify GATE when she learns discrete mathematics."

TABLE 5 **The Truth Table for the Conditional Statement $p \rightarrow q$.**

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

CONVERSE, CONTRAPOSITIVE, AND INVERSE

- The proposition $q \rightarrow p$ is called the **converse** of $p \rightarrow q$.
- The proposition $\neg q \rightarrow \neg p$ is called the **contrapositive**.
- The proposition $\neg p \rightarrow \neg q$ is called the **inverse** of $p \rightarrow q$.
- The contrapositive, $\neg q \rightarrow \neg p$, always has the same truth value as $p \rightarrow q$.
- Neither the converse, $q \rightarrow p$, nor the inverse, $\neg p \rightarrow \neg q$, has the same truth value as $p \rightarrow q$
- When two compound propositions always have the same truth value we call them **equivalent**,
- So that a conditional statement and its contrapositive are equivalent.
- The converse and the inverse of a conditional statement are also equivalent , but neither is equivalent to the original conditional statement.
- **EXAMPLE 6** "The home team wins whenever it is raining?"
- "*q* whenever *p*" is one of the ways to express the conditional statement $p \rightarrow q$,
- The original statement can be rewritten as "If it is raining, then the home team wins."
- <u>Contrapositive</u> : "If the home team does not win, then it is not raining."
- <u>Converse</u> : "If the home team wins, then it is raining."
- <u>Inverse</u> : "If it is not raining, then the home team does not win."
- Only the contrapositive is equivalent to the original statement.

- **DEFINITION 6** Let p and q be propositions. The ***biconditional statement*** $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi - implications*.
- some other common ways to express $p \leftrightarrow q$:
- "p is necessary and sufficient for q"
- "if p then q, and conversely" "p iff q ."
- $p \leftrightarrow q$ has exactly the same truth value as $(p \to q) \wedge (q \to p)$.
- **EXAMPLE 7** Let p : "You can take the flight," and
- q:"You buy a ticket." $p \leftrightarrow q$:
- "You can take the flight if and only if you buy a ticket."
- Connectives : conjunctions, disjunctions, conditional and biconditional statements—as well as negations . We can use these connectives to build up **compound propositions .**
- **EXAMPLE 8** Construct the truth table of the compound proposition $(p \vee \neg q) \to (p \wedge q)$

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

TABLE 7 The Truth Table of $(p \vee \neg q) \to (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \to (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

- Precedence of Logical Operators
- Logic and Bit Operations
- A **bit** is a symbol with two possible values, 0 (zero) and 1 (one).
- 1 represents T (true), 0 represents F (false).
- A variable is called a **Boolean variable** if its value is either true or false, a Boolean variable can be represented using a bit.
- DEFINITION 7 A ***bit string*** is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.
- EXAMPLE 9 101010011 is a bit string of length nine.
- **bitwise *OR***, **bitwise *AND***, and **bitwise *XOR*** use the symbols ∨, ∧, and ⊕ respectively.
- EXAMPLE 10 Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit strings 01 1011 0110 and 11 0001 1101.
- 01 1011 0110
- 11 0001 1101
- 11 1011 1111 bitwise *OR*
- 01 0001 0100 bitwise *AND*
- 10 1010 1011 bitwise *XOR*

TABLE 8 Precedence of Logical Operators.

Operator	*Precedence*
¬	1
∧	2
∨	3
→	4
↔	5

Truth Value	*Bit*
T	1
F	0

TABLE 9 Table for the Bit Operators *OR*, *AND*, and *XOR*.

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

- **Translating English Sentences**
- **EXAMPLE 11** "You can access the Internet from campus only if you are a computer science major or you are not a freshman."
- let *a*- "You can access the Internet from campus," *b*- "You are a computer science major," c-"You are a freshman,"
- This sentence can be represented as $a \rightarrow (b \lor \neg c)$.
- **EXAMPLE 12** "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."
- Let *q*, *r*, and *s* represent "You can ride the roller coaster," "You are under 4 feet tall," and "You are older than 16 years old," respectively.
- Then the sentence can be translated to $(r \land \neg s) \rightarrow \neg q$.
- **EXAMPLE 13** "The automated reply cannot be sent when the file system is full"
- Let *p* denote "The automated reply can be sent" and *q* denote "The file system is full."
- Then the sentence can be represented by the conditional statement $q \rightarrow \neg p$.

1.2 Propositional Equivalences

- **DEFINITION 1** A compound proposition that is always true is called a ***tautology***. A compound proposition that is always false is called a ***contradiction***. A compound proposition that is neither a tautology nor a contradiction is called a ***contingency***.
- **EXAMPLE 1** tautology $p \lor \neg p$ and contradiction $p \land \neg p$
- **Logical Equivalences** Compound propositions that have the same truth values in all possible cases are called **logically equivalent**.
- **DEFINITION 2** The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.
- The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ for logical equivalence.
- **EXAMPLE 2** Show that $\neg(p \lor q)$ and $\neg p \land \neg q$ are logically equivalent

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \lor \neg p$	$p \land \neg p$
T	F	T	F
F	T	T	F

TABLE 2 De Morgan's Laws.

$\neg(p \land q) \equiv \neg p \lor \neg q$

$\neg(p \lor q) \equiv \neg p \land \neg q$

TABLE 3 Truth Tables for $\neg(p \lor q)$ and $\neg p \land \neg q$.

p	q	$p \lor q$	$\neg(p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

- **EXAMPLE 3** Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.
- 2^n rows are required if a compound proposition involves n propositional variables.
- For 3 propositional variable 8 rows.
- For 4 propositional variable 16 rows.
- **EXAMPLE 4** Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the *distributive law* of disjunction over conjunction.

TABLE 4 Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

TABLE 5 A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \rightarrow q \equiv \neg p \vee q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

$p \vee q \equiv \neg p \rightarrow q$

$p \wedge q \equiv \neg(p \rightarrow \neg q)$

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

- De Morgan's laws extend to $\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$ and $\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n)$.
- $\neg\left(\bigvee_{j=1}^{n} p_j\right) \equiv \bigwedge_{j=1}^{n} \neg p_j$ and $\neg\left(\bigwedge_{j=1}^{n} p_j\right) \equiv \bigvee_{j=1}^{n} \neg p_j$.
- **Constructing New Logical Equivalences**
- **EXAMPLE 6** Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.
- $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$
- $\equiv \neg(\neg p) \wedge \neg q$ by the second De Morgan law
- $\equiv \boldsymbol{p \wedge \neg q}$ by the double negation law
- **EXAMPLE 7** Show that ¬(p ∨ (¬p ∧ q)) and ¬p ∧ ¬q are logically equivalent.
- $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ by the second De Morgan law
- $\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$ by the first De Morgan law
- $\equiv \neg p \wedge (p \vee \neg q)$ by the double negation law
- $\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ by the second distributive law
- $\equiv \mathbf{F} \vee (\neg p \wedge \neg q)$ by Negation law
- $\equiv \boldsymbol{\neg p \wedge \neg q}$ by the identity law for **F**

- **EXAMPLE 8** Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.
- $(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ [$p \rightarrow q \equiv \neg p \vee q$]
- $\equiv (\neg p \vee \neg q) \vee (p \vee q)$ by the first De Morgan law
- $\equiv (\neg p \vee p) \vee (\neg q \vee q)$ by the associative and commutative laws for disjunction
- $\equiv \mathbf{T} \vee \mathbf{T} \equiv \mathbf{T}$
- **Propositional Satisfiability**
- A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true.
- When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is **unsatisfiable**.
- A compound proposition is unsatisfiable if and only if its negation is a tautology.
- **EXAMPLE 9**
- Determine whether each of the compound propositions $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$, $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable.
- $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true when the three variable p, q, and r have the same truth value .
- Hence, it is satisfiable as there is at least one assignment of truth values for p, q, and r that makes it true.

- $(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$ is true when at least one of p, q, and r is true and at least one is false.
- Hence, $(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$ is satisfiable,as there is at least one assignment of truth values for p, q, and r that makes it true.
- Finally, $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p) \land (p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$ to be true, $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$ and $(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$ must both be true.
- For the first to be true, the three variables must have the same truth values, and for the second to be true, at least one of three variables must be true and at least one must be false.
- However, these conditions are contradictory.
- From these observations we conclude that no assignment of truth values to p, q, and r makes $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p) \land (p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$ true.
- Hence, it is unsatisfiable.
- ***Contingency → Satisfiable → Tautology***

1.3 Predicates and Quantifiers

- **Predicates**
- Statements involving variables, such as "$x > 3$," "$x = y + 3$," "$x + y = z$,"
- The statement "*x* is greater than 3" has two parts.
- The first part, the variable *x*, is the subject of the statement. The second part—the **predicate**, "is greater than 3".
- We can denote the statement "*x* is greater than 3" by *P* (*x*), where *P* denotes the predicate "is greater than 3" and *x* is the variable.
- The statement *P* (*x*) is also said to be the value of the **propositional function** *P* at *x*.
- Once a value has been assigned to the variable *x*, the statement *P* (*x*) becomes a proposition and has a truth value.
- **EXAMPLE 1** Let *P* (*x*) denote the statement "$x > 3$." What are the truth values of *P* (4) and *P* (2)?
- *Solution:* *P* (4), which is the statement "$4 > 3$," is true.
- *P* (2), which is the statement "$2 > 3$," is false.
- **EXAMPLE 2** Let *Q*(*x*, *y*) denote the statement "$x = y + 3$." What are the truth values of the propositions *Q*(1, 2) and *Q*(3, 0)?
- *Solution:* *Q*(1, 2) is the statement "$1 = 2 + 3$," which is false.
- *Q*(3, 0) is the proposition "$3 = 0 + 3$," which is true.
- PRECONDITIONS AND POSTCONDITIONS
- Predicates are also used to establish the correctness of computer programs.
- The statements that describe valid input are known as **preconditions**

- The conditions that the output should satisfy when the program has run are known as **postconditions**.
- **EXAMPLE 3** Consider the following program, designed to interchange the values of two variables *x* and *y*.
- `temp := x`
- `x := y`
- `y := temp`
- Find predicates that we can use as the precondition and the postcondition to verify the correctness of this program.
- *Solution:* **F**or this precondition we can use the predicate *P (x, y)*, where *P (x, y)* is the statement "*x* = *a* and *y* = *b*," where *a* and *b* are the values of *x* and *y* before we run the program.
- Because we want to verify that the program swaps the values of *x* and *y* for all input values, for the postcondition we can use *Q(x, y)*, where *Q(x, y)* is the statement "*x* = *b* and *y* = *a*."
- **Quantifiers**
- Quantification expresses the extent to which a predicate is true over a range of elements.
- In English, the words *all*, *some*,*many*, *none*, and *few* are used in quantifications.
- Two types of quantification here: **Universal Quantification**, which tells us that a predicate is true for every element under consideration,

- And **Existential Quantification**, which tells us that there is one or more element under consideration for which the predicate is true.
- The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.
- THE UNIVERSAL QUANTIFIER
- Many mathematical statements is true for all values of a variable in a particular domain, called the **domain of discourse** (or the **universe of discourse**), often just referred to as the **domain**.
- **DEFINITION 1** The *universal quantification* of $P(x)$ is the statement "$P(x)$ for all values of x in the domain." The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier.** We read $\forall x P(x)$ as "for all x $P(x)$" or "for every x $P(x)$."
- An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$.
- Besides "for all" and "for every," universal quantification can be expressed in many other ways, including "all of," "for each," "given any," "for arbitrary," "for each,".
- **EXAMPLE 4** Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$,where the domain consists of all real numbers?
- *Solution:* Because $P(x)$ is true for all real numbers x, $\forall x P(x)$ is true.
- **EXAMPLE 5** Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?
- *Solution:* $Q(x)$ is not true for every real number x,

- because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample. Thus $\forall x Q(x)$ is false.
- When all the elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$—it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction $P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$,
- Because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.
- **EXAMPLE 6** What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?
- *Solution:* The statement $\forall x P(x)$ is the same as the conjunction $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$, because the domain consists of the integers 1, 2, 3, and 4.
- Because $P(4)$, "$4^2 < 10$," is false, it follows that $\forall x P(x)$ is false.
- THE EXISTENTIAL QUANTIFIER
- **DEFINITION 2** The *existential quantification* of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$." We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the *existential quantifier*.
- Besides the phrase "there exists," we can also express $\exists$ in many other ways, such as by using the words "for some," "for at least one," or "there is."
- The existential quantification $\exists x P(x)$ is read as "There is an x such that $P(x)$,"
- "There is at least one x such that $P(x)$,"
- or "For some $x P(x)$."

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

- **EXAMPLE 7** Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists xP(x)$,where the domain consists of all real numbers?
- *Solution:* Because "$x > 3$" is sometimes true—for instance, when $x = 4$—the existential quantification of $P(x)$, which is $\exists xP(x)$, is true.
- **EXAMPLE 8** Let $Q(x)$ denote the statement "$x = x + 1$."What is the truth value of the quantification$\exists xQ(x)$,where the domain consists of all real numbers?
- *Solution:* Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists xQ(x)$, is false.
- When all elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$— the existential quantification $\exists xP(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$,
- Because this disjunction is true if and only if at least one of $P(x_1)$, $P(x_2)$,.., $P(x_n)$ is true.
- **EXAMPLE 9** What is the truth value of $\exists xP(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?
- *Solution:* Because the domain is {1, 2, 3, 4}, the proposition $\exists xP(x)$ is the same as the disjunction $P(1) \vee P(2) \vee P(3) \vee P(4)$.
- Because $P(4)$, "$4^2 > 10$," is true, it follows that $\exists xP(x)$ is true.

- THE UNIQUENESS QUANTIFIER the **uniqueness quantifier**, denoted by ∃! or ∃1.
- The notation $\exists! x P(x)$ [or $\exists 1 x P(x)$] states "There exists a unique x such that $P(x)$ is true."
- (Other phrases include "there is exactly one" and "there is one and only one.")
- For instance, $\exists! x(x - 1 = 0)$, where the domain is the set of real numbers, states that there is a unique real number x such that $x - 1 = 0$. This is a true statement, as $x = 1$ is the unique real number such that $x - 1 = 0$.
- **Quantifiers with Restricted Domains**
- **EXAMPLE 10** What do the statements $\forall x < 0\ (x^2 > 0)$, $\forall y \neq 0\ (y^3 \neq 0)$, and $\exists z > 0\ (z^2 = 2)$ mean, where the domain in each case consists of the real numbers?
- *Solution:* The statement $\forall x < 0\ (x^2 > 0)$ states that for every real number x with $x < 0$, $x^2 > 0$.
- "The square of a negative real number is positive." This statement is same as $\forall x(x < 0 \rightarrow x^2 > 0)$.
- The statement $\forall y \neq 0\ (y^3 \neq 0)$ states that for every real number y with $y \neq 0$, we have $y^3 \neq 0$.
- "The cube of every nonzero real number is nonzero." this statement is same as $\forall y(y = 0 \rightarrow y^3 = 0)$.
- The statement $\exists z > 0\ (z^2 = 2)$ states that there exists a real number z with $z > 0$ such that $z^2 = 2$.
- "There is a positive square root of 2." This statement is equivalent to $\exists z(z > 0 \wedge z^2 = 2)$.
- **Precedence of Quantifiers**
- The quantifiers ∀ and ∃ have higher precedence than all logical operators from propositional calculus. For example, $\forall x P(x) \vee Q(x)$ means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x(P(x) \vee Q(x))$.
- **Binding Variables**
- When a quantifier is used variable x, we say that this occurrence of the variable is **bound**.

- An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**.
- The part of a logical expression to which a quantifier is applied is called the **scope** of this quantifier.
- Consequently, a variable is free if it is outside the scope of all quantifiers in the formula that specify this variable.
- **EXAMPLE 11** In the statement $\exists x(x + y = 1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to this variable.
- In the statement $\exists x(P(x) \land Q(x)) \lor \forall xR(x)$, all variables are bound. The scope of the first quantifier, $\exists x$, is the expression $P(x) \land Q(x)$ because $\exists x$ is applied only to $P(x) \land Q(x)$, and not to the rest of the statement.
- The scope of the second quantifier, $\forall x$, is the expression $R(x)$.
- **Logical Equivalences Involving Quantifiers**
- **DEFINITION 3** Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates which domain of discourse is used for the variables.
- We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.
- **EXAMPLE 12** $\forall x(P(x) \land Q(x)) \equiv \forall xP(x) \land \forall xQ(x)$

- **Negating Quantified Expressions**
- For instance, consider the negation of the statement "Every student in your class has taken a course in calculus." This statement is a universal quantification, namely, $\forall x P(x)$,
- Where $P(x)$:"x has taken a course in calculus" and the domain :students in your class.
- The negation of this statement is "It is not the case that every student in your class has taken a course in calculus."
- This is equivalent to "There is a student in your class who has not taken a course in calculus."
- And this is existential quantification of the negation of the original propositional function, namely, $\exists x \neg P(x)$.
- This example illustrates the following logical equivalence: $\neg \forall x P(x) \equiv \exists x \neg P(x)$
- For instance, "There is a student in this class who has taken a course in calculus."
- This is the existential quantification $\exists x Q(x)$,
- Where $Q(x)$ is the statement "x has taken a course in calculus."
- The negation of this statement is the proposition "It is not the case that there is a student in this class who has taken a course in calculus."
- This is equivalent to "Every student in this class has not taken calculus,"
- Which is the universal quantification of the negation of the original propositional function, or, phrased in the language of quantifiers , $\forall x \neg Q(x)$.

- This example illustrates the equivalence $\neg\exists x Q(x) \equiv \forall x \neg Q(x)$
- **EXAMPLE 13** What are the negations of the statements "There is an honest politician"?
- *Solution:* Let $H(x)$ denote "x is honest." Then the statement is represented by $\exists x H(x)$, where the domain consists of all politicians. $\neg\exists x H(x) \equiv \forall x \neg H(x)$.
- This negation can be expressed as "Every politician is dishonest.", this statement often means "Not all politicians are honest."
- **EXAMPLE 14** What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?
- *Solution:* $\neg\forall x(x^2 > x), \equiv \exists x \neg(x^2 > x)$,This can be rewritten as $\exists x(x^2 \leq x)$.
- $\neg\exists x(x^2 = 2) \equiv \forall x \neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$.
- The truth values of these statements depend on the domain.
- **EXAMPLE 15** Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.
- *Solution:* $\neg\forall x(P(x) \rightarrow Q(x)) \equiv \exists x(\neg(P(x) \rightarrow Q(x)))$
- We know that $\neg(P(x) \rightarrow Q(x))$ and $P(x) \wedge \neg Q(x)$ are logically equivalent for every x.
- $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

TABLE 2 **De Morgan's Laws for Quantifiers.**

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Translating from English into Logical Expressions

- **EXAMPLE 16** Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.
- *Solution:* First, rewrite : "For every student in this class, that student has studied calculus."
- Next,introduce a variable *x* : "For every student *x* in this class, *x* has studied calculus."
- $C(x)$: "*x* has studied calculus."
- If the domain for *x* consists of the students in the class, we can translate as $\forall x C(x)$.
- If we change the domain to consist of all people "For every person *x*,if person *x* is a student in this class then *x* has studied calculus."
- $S(x)$: " person *x* is in this class",
- Our statement can be expressed as $\forall x(S(x) \rightarrow C(x))$.
- [*Caution!* Our statement *cannot* be expressed as $\forall x(S(x) \wedge C(x))$ because this statement says that all people are students in this class and have studied calculus!]
- Finally, two-variable quantifier $Q(x, y)$ for the statement "student *x* has studied subject *y*."
- Then we would replace $C(x)$ by $Q(x, \text{calculus})$ in both approaches to obtain $\forall x Q(x, \text{calculus})$ or $\forall x(S(x) \rightarrow Q(x, \text{calculus}))$.
- **EXAMPLE 17** Use predicates and quantifiers to express the system specifications "Every mail message larger than one megabyte will be compressed" and "If a user is active, at least one network link will be available."
- *Solution:* Let $S(m, y)$ be "Mail message *m* is larger than *y* megabytes," where *x* has the domain of all mail messages and *y* is a positive real number, and let $C(m)$ denote "Mail message *m* will be compressed."

- Then $\forall m(S(m, 1) \rightarrow C(m))$.
- Let $A(u)$ represent "User u is active," where the variable u has the domain of all users,
- let $S(n, x)$ denote "Network link n is in state x," where n has the domain of all network links and x has the domain of all possible states for a network link.
- "If a user is active, at least one network link will be available."
- Then $\exists u A(u) \rightarrow \exists n S(n, \text{available})$.
- **EXAMPLE 18** Consider these statements. The first two are called *premises* and the third is called the *conclusion*.The entire set is called an *argument*.
- "All lions are fierce."
- "Some lions do not drink coffee."
- "Some fierce creatures do not drink coffee."
- Let $P(x)$: "x is a lion," $Q(x)$: "x is fierce," and $R(x)$: "x drinks coffee," be the statements.
- Assuming that the domain consists of all creatures, express the statements in the argument using quantifiers and P (x), Q(x), and R(x).
- *Solution:* We can express these statements as:
- $\forall x(P(x) \rightarrow Q(x))$.
- $\exists x(P(x) \wedge \neg R(x))$.
- $\exists x(Q(x) \wedge \neg R(x))$.
- The second statement cannot be written as $\exists x(P(x) \rightarrow \neg R(x))$.

- The reason is that $P(x) \rightarrow \neg R(x)$ is true whenever x is not a lion,So that $\exists x(P(x) \rightarrow \neg R(x))$ is true as long as there is at least one creature that is not a lion, even if every lion drinks coffee.
- Similarly, the third statement cannot be written as $\exists x(Q(x) \rightarrow \neg R(x))$.
- **EXAMPLE 19** Consider these statements, of which the first three are premises and the fourth is a valid conclusion.
- "All hummingbirds are richly colored."
- "No large birds live on honey."
- "Birds that do not live on honey are dull in color."
- "Hummingbirds are small."
- Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a hummingbird," "x is large," "x lives on honey," and "x is richly colored," respectively.
- Assuming that the domain consists of all birds,express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.
- *Solution:* We can express the statements in the argument as
- $\forall x(P(x) \rightarrow S(x))$.
- $\neg \exists x(Q(x) \wedge R(x))$.
- $\forall x(\neg R(x) \rightarrow \neg S(x))$.
- $\forall x(P(x) \rightarrow \neg Q(x))$.

1.4 Nested Quantifiers

- **Nested quantifiers**, where one quantifier is within the scope of another, such as $\forall x \exists y(x + y = 0)$ is the same thing as $\forall x Q(x)$, where $Q(x)$ is $\exists y P(x, y)$, where $P(x, y)$ is $x + y = 0$.
- **EXAMPLE 1** Assume that the domain for the variables x and y consists of all real numbers.
- The statement $\forall x \forall y(x + y = y + x)$ says that $x + y = y + x$ for all real numbers x and y. This is the commutative law for addition of real numbers.
- $\forall x \exists y(x + y = 0)$ says that for every real number x ,y such that $x + y = 0$. This states that every real number has an additive inverse.
- Similarly, $\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$ is the associative law for addition of real numbers.
- **EXAMPLE 2** Translate into English the statement $\forall x \forall y((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$, where the domain for both variables consists of all real numbers.
- *Solution:* for real numbers x and y, if x is positive and y is negative, then xy is negative.
- **The Order of Quantifiers**
- The order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.
- **EXAMPLE 3** Let $P(x, y)$ be the statement "$x + y = y + x$."
- What are the truth values of the quantifications $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$
- where the domain for all variables consists of all real numbers?

- *Solution:* $\forall x \forall y P(x, y)$ denotes the proposition "For all real numbers x, y, $x + y = y + x$."
- Because P (x, y) is true for all real numbers x and y (it is the commutative law for addition), the proposition $\forall x \forall y P(x, y)$ is true.
- The statement $\forall y \forall x P(x, y)$ says "For all real numbers y, x, x + y = y + x."
- That is, $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ have the same meaning,and both are true.
- **EXAMPLE 4** Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?
- *Solution:* $\exists y \forall x Q(x, y)$ "For every real number x there is a real number y such that $Q(x, y)$."
- No matter what value of y is chosen, there is only one value of x for which $x + y = 0$. Because there is no real number y such that $x + y = 0$ for all real numbers x, $\exists y \forall x Q(x, y)$ is false.
- $\forall x \exists y Q(x, y)$"For every real number x there is a real number y such that $Q(x, y)$."
- Given a real number x, there is a real number y such that $x + y = 0$; namely, $y = -x$. Hence,the statement $\forall x \exists y Q(x, y)$ is true.
- The statements $\exists y \forall x P(x, y)$ and $\forall x \exists y P(x, y)$ are not logically equivalent.
- In the second case, y depends on x, whereas in the first case, y is a constant independent of x.

- If $\exists y \forall x P(x, y)$ is true, then $\forall x \exists y P(x, y)$ must also be true.
- However, if $\forall x \exists y P(x, y)$ is true, it is not necessary for $\exists y \forall x P(x, y)$ to be true.
- **EXAMPLE 5** Let $Q(x, y, z)$ be the statement "$x + y = z$." What are the truth values of the statements $\forall x \forall y \exists z Q(x, y, z)$ and $\exists z \forall x \forall y Q(x, y, z)$, where the domain of all variables consists of all real numbers?
- *Solution:* $\forall x \forall y \exists z Q(x, y, z)$, "For all real numbers x and y there is a real number z such that $x + y = z$," is true.
- $\exists z \forall x \forall y Q(x, y, z)$, "There is a real number z such that for all real numbers x and y, $x + y = z$," is false, Because there is no value of z that satisfies the equation $x + y = z$ for all values of x and y.

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

- **Translating Mathematical Statements into Statements Involving Nested Quantifiers**
- **EXAMPLE 6** Translate the statement "The sum of two positive integers is always positive" into a logical expression.
- *Solution:* First rewrite : "For every two integers, if these integers are both positive , then the sum of these integers is positive."
- Introduce the variables x and y to obtain "For all positive integers x and y, $x + y$ is positive."
- $\forall x \forall y((x > 0) \land (y > 0) \rightarrow (x + y > 0))$,where the domain consists of all integers.
- Note that we could also translate this using the positive integers as the domain.
- The statement becomes "For every two positive integers, the sum of these integers is positive.
- $\forall x \forall y(x + y > 0)$,where the domain for both variables consists of all positive integers.
- **EXAMPLE 7** Translate the statement "Every real number except zero has a multiplicative inverse."
- *Solution:* We first rewrite this as "For every real number x, if $x \neq 0$, then there exists a real number y such that $xy = 1$."
- This can be rewritten as $\forall x((x \neq 0) \rightarrow \exists y(xy = 1))$.
- **Translating from Nested Quantifiers into English**
- The first step in translating such an expression is to write out what the quantifiers and predicates in the expression mean.
- The next step is to express this meaning in a simpler sentence.

- **EXAMPLE 8** Translate the statement $\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$ into English, where $C(x)$ is "x has a computer," $F(x, y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.
- *Solution:* The statement says that "for every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends".
- Or, "every student in your school has a computer or has a friend who has a computer".
- **EXAMPLE 9** Translate the statement $\exists x \forall y \forall z((F(x, y) \wedge F(x, z) \wedge (y \neq z)) \rightarrow \neg F(y,z))$ into English, where $F(a,b)$ means a and b are friends and the domain for x, y, and z consists of all students in your school.
- *Solution:* It follows that "there is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends, then y and z are not friends".
- Or, there is a student none of whose friends are also friends with each other.
- **Translating English Sentences into Logical Expressions**
- **EXAMPLE 10** Express the statement "If a person is female and is a parent , then this person is someone's mother" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.
- *Solution:* The statement can be expressed as "For every person x, if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y."

- $F(x)$:"x is female," $P(x)$:"x is a parent," and $M(x, y)$:"x is the mother of y."
- The original statement can be represented as $\forall x((F(x) \wedge P(x)) \rightarrow \exists y M(x, y))$.
- We can move $\exists y$ to the left, because y does not appear in $F(x) \wedge P(x)$.
- The logically equivalent expression $\forall x \exists y((F(x) \wedge P(x)) \rightarrow M(x, y))$.
- **EXAMPLE 11** Express the statement "Everyone has exactly one best friend" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.
- *Solution:* First rewrite : "For every person x, person x has exactly one best friend."
- Introducing the universal quantifier, "$\forall x$(person x has exactly one best friend)," where the domain consists of all people.
- To say that x has exactly one best friend means that there is a person y who is the best friend of x, and furthermore, that for every person z, if person z is not person y, then z is not the best friend of x.
- When $B(x, y)$ to be the statement "y is the best friend of x,"
- The statement can be expressed as $\forall x \exists y(B(x, y) \wedge \forall z((z \neq y) \rightarrow \neg B(x, z)))$.
- We can write this statement as $\forall x \exists! y B(x, y)$, where $\exists!$ is the "uniqueness quantifier"

- **EXAMPLE 12** Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."
- *Solution:* Let $P(w, f)$ be "w has taken f" and $Q(f, a)$ be "f is a flight on a."
- We can express the statement as $\exists w \forall a \exists f (P(w, f) \land Q(f, a))$,
- Where the domains of discourse for w, f, and a consist of all the women in the world, all airplane flights, and all airlines, respectively.
- It could also be expressed as $\exists w \forall a \exists f\, R(w, f, a)$, Where $R(w, f, a)$ is "w has taken f on a."
- **Negating Nested Quantifiers**
- **EXAMPLE 13** Express the negation of the statement $\forall x \exists y(xy = 1)$ so that no negation precedes a quantifier.
- *Solution:* $\neg \forall x \exists y(xy = 1) \equiv \exists x \neg \exists y(xy = 1) \equiv \exists x \forall y \neg (xy = 1) \equiv \exists x \forall y(xy \neq 1)$.
- **EXAMPLE 14** Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."
- *Solution:* $\neg \exists w \forall a \exists f (P(w, f) \land Q(f, a))$, [$P(w,f)$ is "w has taken f", $Q(f,a)$ is "f is a flight on a."]
- $\neg \exists w \forall a \exists f (P(w, f) \land Q(f, a)) \equiv \forall w \neg \forall a \exists f (P(w, f) \land Q(f, a))$
- $\equiv \forall w \exists a \neg \exists f (P(w, f) \land Q(f, a)) \equiv \forall w \exists a \forall f \neg (P(w, f) \land Q(f, a))$
- $\equiv \forall w \exists a \forall f (\neg P(w, f) \lor \neg Q(f, a))$.
- "For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline."

1.5 Rules of Inference

- **DEFINITION 1** An ***argument*** in propositional logic is a sequence of propositions. All but the final proposition in the argument are called ***premises*** and the final proposition is called the ***conclusion***.
- An argument is ***valid*** if the truth of all its premises implies that the conclusion is true.
- An argument form is *valid*, the conclusion is true if the premises are all true.
- The argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion q is valid, when $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$ is a tautology.
- We can first establish the validity of some relatively simple argument forms, called **rules of inference**.
- These rules of inference can be used as building blocks to construct more complicated valid argument forms.
- The tautology $(p \wedge (p \rightarrow q)) \rightarrow q$ is the basis of the rule of inference called **modus ponens**, or the **law of detachment**. (where, the symbol ∴ denotes "therefore"):
- p
- $\underline{P \rightarrow q}$
- $\therefore q$
- **EXAMPLE 1** Suppose that the conditional statement "If it snows today, then we will go skiing" and its hypothesis, "It is snowing today," are true.
- Then, by modus ponens, it follows that the conclusion of the conditional statement, "We will go skiing," is true.

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

- **EXAMPLE 2** State which rule of inference is the basis of the following argument: "It is below freezing now.Therefore, it is either below freezing or raining now."
- *Solution:* Let p : "It is below freezing now" and q : "It is raining now."
- $\underline{\quad P \quad}$
- $\therefore p \lor q$
- This is an argument that uses the addition rule.

- **EXAMPLE 3** Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."
- *Solution:* Let p : "It is sunny this afternoon," q : "It is colder than yesterday," r : "We will go swimming," s : "We will take a canoe trip," and t :"We will be home by sunset."
- Then the premises become $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$. The conclusion is simply t.

	Step	Reason
1.	$\neg p \wedge q$	Premise
2.	$\neg p$	Simplification using (1)
3.	$r \rightarrow p$	Premise
4.	$\neg r$	Modus tollens using (2) and (3)
5.	$\neg r \rightarrow s$	Premise
6.	s	Modus ponens using (4) and (5)
7.	$s \rightarrow t$	Premise
8.	t	Modus ponens using (6) and (7)

- In truth table due to 5 propositional variables $p,q,r,s,$and t would have 32 rows.

- **EXAMPLE 4** Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."
- *Solution:* Let p :"You send me an e-mail message," q : "I will finish writing the program," r: "I will go to sleep early," and s :"I will wake up feeling refreshed."
- Then the premises are $p \to q$, $\neg p \to r$, and $r \to s$. conclusion is $\neg q \to s$.

	Step	Reason
1.	$p \to q$	Premise
2.	$\neg q \to \neg p$	Contrapositive of (1)
3.	$\neg p \to r$	Premise
4.	$\neg q \to r$	Hypothetical syllogism using (2) and (3)
5.	$r \to s$	Premise
6.	$\neg q \to s$	Hypothetical syllogism using (4) and (5)

- **Resolution**
- Computer programs have been developed to automate the task of reasoning and proving theorems.
- Many of these programs make use of a rule of inference known as **resolution**.
- This rule of inference is based on the tautology $((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$.

- The final disjunction in the resolution rule, $q \lor r$, is called the **resolvent**.
- **EXAMPLE 5** Use resolution to show that the hypotheses "Jasmine is skiing or it is not snowing" and "It is snowing or Bart is playing hockey" imply that "Jasmine is skiing or Bart is playing hockey."
- *Solution:* Let p :"It is snowing," q : "Jasmine is skiing," and r : "Bart is playing hockey."
- We can represent the hypotheses as $\neg p \lor q$ and $p \lor r$,respectively.
- Using resolution, the proposition $q \lor r$, "Jasmine is skiing or Bart is playing hockey," follows.
- The hypotheses and the conclusion must be expressed as **clauses**, where a clause is a disjunction of variables or negations of these variables.
- **EXAMPLE 6** Show that the premises $(p \land q) \lor r$ and $r \rightarrow s$ imply the conclusion $p \lor s$.
- *Solution:*

1. $(p \land q) \lor r \equiv (p \lor r) \land (q \lor r)$. [Distributive laws]
2. $r \rightarrow s \equiv \neg r \lor s$.

- Using the two clauses $p \lor r$ and $\neg r \lor s$, we can use resolution to conclude $p \lor s$.
- **Fallacies**
- Several common fallacies arise in incorrect arguments. These fallacies resemble rules of inference, but are based on contingencies rather than tautologies.
- $((p \rightarrow q) \land q) \rightarrow p$ is not a tautology, because it is false when p is false and q is true.

- This type of incorrect reasoning is called the **fallacy of affirming the conclusion**.
- **EXAMPLE 7** Is the following argument valid?
- If you do every problem in this book, then you will learn discrete mathematics . You learned discrete mathematics. Therefore, you did every problem in this book.
- *Solution:* Let *p* :"You did every problem in this book.",*q* :"You learned discrete mathematics."
- Then this argument is of the form: $((p \rightarrow q) \land q) \rightarrow p$.
- This is an example of an incorrect argument using the fallacy of affirming the conclusion.
- It is possible for you to learn discrete mathematics in some way other than by doing every problem in this book.
- $((p \rightarrow q) \land \neg p) \rightarrow \neg q$ is not a tautology, because it is false when *p* is false and *q* is true.
- Many incorrect arguments use this incorrectly as a rule of inference. This type of incorrect reasoning is called the **fallacy of denying the hypothesis**.
- **EXAMPLE 8** Let *p* and *q* be as in Example 7.
- If the conditional statement $p \rightarrow q$ is true, and $\neg p$ is true,is it correct to conclude that $\neg q$ is true? In other words, is it correct to assume that you did not learn discrete mathematics if you did not do every problem in the book, assuming that if you do every problem in this book, then you will learn discrete mathematics?

- *Solution*: It is possible that you learned discrete mathematics even if you did not do every problem in this book.
- This incorrect argument is of the form $p \to q$ and $\neg p$ imply $\neg q$, which is an example of the fallacy of denying the hypothesis.
- **Rules of Inference for Quantified Statements**

TABLE 2 **Rules of Inference for Quantified Statements.**

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

- **EXAMPLE 9** Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."
- *Solution:*
- Let $D(x)$:"x is in this discrete mathematics class," $C(x)$: "x has taken a course in computer science." The premises are $\forall x(D(x) \to C(x))$ and D(Marla).The conclusion is C(Marla).
- **Step** | **Reason**

1. $\forall x(D(x) \to C(x))$ — Premise
2. $D(\text{Marla}) \to C(\text{Marla})$ — Universal instantiation from (1)

3. D(Marla) Premise
4. C(Marla) Modus ponens from (2) and (3)

- **EXAMPLE 10** Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."
- *Solution:* Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam."
- The premises are $\exists x(C(x) \land \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \land \neg B(x))$.
- **Step** **Reason**

Step		Reason
1.	$\exists x(C(x) \land \neg B(x))$	Premise
2.	$C(a) \land \neg B(a)$	Existential instantiation from (1)
3.	$C(a)$	Simplification from (2)
4.	$\forall x(C(x) \rightarrow P(x))$	Premise
5.	$C(a) \rightarrow P(a)$	Universal instantiation from (4)
6.	$P(a)$	Modus ponens from (3) and (5)
7.	$\neg B(a)$	Simplification from (2)
8.	$P(a) \land \neg B(a)$	Conjunction from (6) and (7)
9.	$\exists x(P(x) \land \neg B(x))$	Existential generalization from (8)

- **Combining Rules of Inference for Propositions and Quantified Statements**
- Universal instantiation and modus ponens are used so often together, this combination of rules is sometimes called **universal modus ponens**.
- $\forall x(P(x) \rightarrow Q(x))$
- $P(a)$, where a is a particular element in the domain
- $\therefore Q(a)$
- **Universal modus tollens** combines universal instantiation and modus tollens and can be expressed in the following way:
- $\forall x(P(x) \rightarrow Q(x))$
- $\neg Q(a)$, where a is a particular element in the domain
- $\therefore \neg P(a)$
- **EXAMPLE 11** Assume that "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" is true.Use universal modus ponens to show that $100^2 < 2^{100}$.
- *Solution:* Let $P(n)$ denote "$n > 4$" and $Q(n)$ denote "$n^2 < 2^n$."
- $\forall n(P(n) \rightarrow Q(n))$,where the domain consists of all positive integers.
- $P(100)$ is true because $100 > 4$.
- It follows by universal modus ponens that $Q(100)$ is true, $100^2 < 2^{100}$.

1.6 Introduction to Proofs

- A proof is a valid argument that establishes the truth of a mathematical statement.
- **Some Terminology**
- A **theorem** is a statement that can be shown to be true.Less important theorems sometimes are called **propositions**. (Theorems can also be referred to as **facts** or **results**.)
- A **proof** is a valid argument that establishes the truth of a theorem.
- The statements used in a proof can include **axioms** (or **postulates**), which are statements we assume to be true.
- A less important theorem that is helpful in the proof of other results is called a **lemma.**
- A **corollary** is a theorem that can be established directly from a theorem that has been proved.
- A **conjecture** is a statement that is being proposed to be a true statement.
- Many times conjectures are shown to be false, so they are not theorems.
- **Methods of Proving :Direct Proofs**
- A **direct proof** shows that a conditional statement $p \rightarrow q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs.
- In a direct proof, we assume that p is true and use axioms,definitions, and previously proven theorems, together with rules of inference, to show that q must also be true.

- **DEFINITION 1**
- The integer n is *even* if there exists an integer k such that $n = 2k$, and n is *odd* if there exists an integer k such that $n = 2k + 1$. Two integers have the *same parity* when both are even or both are odd; they have *opposite parity* when one is even and the other is odd.
- **EXAMPLE 1** Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."
- *Solution:* $\forall n(P(n) \rightarrow Q(n))$, where $P(n)$ is "n is an odd integer" and $Q(n)$ is "n^2 is odd."
- By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer.
- We want to show that n^2 is also odd. We can square both sides of the equation
- $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
- By the definition of an odd integer, we can conclude that n^2 is an odd integer.
- **EXAMPLE 2** Give a direct proof that if m and n are both perfect squares, then mn is also a perfect square.(An integer a is a **perfect square** if there is an integer b such that $a = b^2$.)
- *Solution:* There are integers s and t such that $m = s^2$ and $n = t^2$.
- This tells us that $mn = s^2t^2$.
- $mn =(ss)(tt) = (st)(st) = (st)^2$, using commutativity and associativity of multiplication.
- By the definition of perfect square, it follows that mn is also a perfect square, because it is the square of st, which is an integer.
- **Proof by Contraposition**
- Proofs that do not start with the premises and end with the conclusion, are called **indirect proofs**.

- An useful type of indirect proof is known as **proof by contraposition**.
- The conditional statement $p \to q$ can be proved by showing that its contrapositive, $\neg q \to \neg p$, is true.
- In a proof by contraposition of $p \to q$, we take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\neg p$ must follow.
- **EXAMPLE 3** Prove that if n is an integer and $3n + 2$ is odd, then n is odd.
- *Solution:* Direct proof, we first assume that $3n + 2$ is an odd integer.
- This means that $3n + 2 = 2k + 1$ for some integer k.
- $3n + 1 = 2k$, but there does not seem to be any direct way to conclude that n is odd.
- Proof by Contraposition.
- The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $3n + 2$ is odd, then n is odd" is false; namely, assume that n is even.
- Then, by the definition of an even integer, $n = 2k$ for some integer k.
- Substituting $2k$ for n, $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$.
- This tells us that $3n + 2$ is even , and therefore not odd.
- This is the negation of the premise of the theorem.
- We have proved the theorem "If $3n + 2$ is odd, then n is odd."

- VACUOUS AND TRIVIAL PROOFS A conditional statement $p \to q$ is true when we know that p is false.
- Consequently, if we can show that p is false, then we have a proof, called a **vacuous proof**, of the conditional statement $p \to q$.
- By showing that q is true, it follows that $p \to q$ must also be true. A proof of $p \to q$ that uses the fact that q is true is called a **trivial proof**.
- **DEFINITION 2** The real number r is *rational* if there exist integers p and q with $q \neq 0$ such that $r = p/q$. A real number that is not rational is called *irrational*.
- **Proofs by Contradiction**
- Suppose we want to prove that a statement p is true. Furthermore, suppose that we can find a contradiction q such that $\neg p \to q$ is true.
- Because q is false, but $\neg p \to q$ is true, we can conclude that $\neg p$ is false, which means that p is true.
- Proofs of this type are called **proofs by contradiction**.
- Because a proof by contradiction does not prove a result directly, it is another type of indirect proof.